The dynamics of numerical methods for initial value problems

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The Dynamics of Numerical Methods for Initial Value Problems

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Abstract

Most physical systems are inherently non–linear and very few such systems admit a closed form of solution. In the past 3–4 decades this has led to the numerical investigation of dynamical systems with the aid of computers. The validity of such simulations depends on the the system and the numerical scheme being in some sense close. In recent years various authors have shown that arbitrary convergent numerical schemes will not necessarily have the same dynamic properties as the underlying system. The philosophy behind this thesis is to seek numerical schemes with the same dynamical properties as the system to be approximated.

The thesis contains three contributions to the analysis of dynamical systems and numerics for dynamical systems.

A useful tool in the analysis of a dynamical system are the Lyapunov exponents and we present in this thesis new numerical schemes for their estimation and prove convergence.

The complex Ginzburg–Landau equation arises in a large range of scientific fields: from the study of superconductivity through to turbulent fluid flow and chemical reactions. We consider in detail the dynamic properties of certain finite difference approximations to this equation. Upper–semicontinuity of the semi–discrete global attractor to the true global attractor is proved and numerical results are presented.

We conclude with some results on the structure of the global attractor for the Ginzburg–Landau equation and examine heteroclinic connections.

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Chapter 1

Introduction

1.1 General Introduction

Dynamical systems are found in every branch of science. They provide a mathematical description through equations of physical or "real world" processes which evolve through time. These are termed *deterministic systems* since the future state of a dynamical system is completely determined by the starting or initial condition chosen. In this thesis we will be particularly concerned with the long–time behaviour of dynamical systems. Physically this corresponds to asking what will be the "permanent state" of a given initial condition, ignoring any transient behaviour. This permanent state is said to be an *attractor* for the initial condition. We note that the evolution of a dynamical system may either be continuous or discrete in time.

For example, consider a simple pendulum set in motion by a particular force F_1 and suppose that after a certain time the pendulum comes to rest. Then the state of rest is the attractor for that particular initial condition. Suppose that for another initial force F_2 the pendulum swings periodically for ever, then the periodic orbit is the attractor for the initial condition F_2. The pendulum is an example of a continuous dynamical system; an example of a discrete dynamical system is the evolution of a population measured annually.

Suppose we were given a dynamical system and an initial condition and were asked to find the solution at time t_1 or to determine the long–time evolution. Then for most cases it is either difficult or impossible to find an exact analytic solution. This led to the numerical approximation of dynamical systems to find an approximate solution at time t_1 or the approximate long–time behaviour of solutions. This numerical approximation of a continuous dynamical system will be the most common example of a discrete dynamical system in this thesis. Our philosophy is to seek numerical schemes which have the same qualitative dynamical behaviour as the system they are approximating.

Life would be very dull if dynamical systems were predictable and so fortunately attractors can be complicated objects. In the 19^{th} Century and early 20^{th} Century it was generally thought that, since the evolution of dynamical systems is determined by equations, complex phenomena like turbulence would be predictable. However, as Poincaré proposed and Lorenz discovered in 1963 [92], computing power alone is not enough to resolve unpredictability and certain deterministic systems are inherently "chaotic". Indeed the system studied by Lorenz is the classical example of a continuous chaotic system, now known as the Lorenz equations [92].

The remainder of this first chapter is devoted to introducing a selection of standard dynamical systems definitions and results that might be found in any of the standard text books on dynamical systems such as Hirsch and Smale [74], Guckenheimer and Holmes [59], Temam [128] or Wiggins [133].

We pay particular attention to the description of asymptotic dynamics. In section 1.2 we define the concepts of *absorbing balls* and *global attractors* and state a theorem first proved by [15] which links them. Absorbing balls and global attractors have been considered by many authors for discrete and continuous systems, see for example [63], Ladyzhenskaya [88] or Temam [128]. There has been a considerable amount of work recently showing that if a continuous system has absorbing balls and global attractor then so do certain numerical approximations. In finite dimensions, i.e. for systems in $\mathbb{R}^p$, this question has been considered for Runge–Kutta methods by [78] and linear multi–step methods by [73]. For the approximation of infinite dimensional systems results are much more on a case by case basis. The reduction of reaction–diffusion equation to a system of ordinary differential equations by a finite element method or Galerkin method was considered by [63, 64]. The finite difference approximations to semi–linear parabolic equations were treated in [46]. Finite difference approximations were also considered in this context by [47, 50] for the Kuromoto–Sivashinsky equation. Similarly Yan [138, 137] proves the existence of a discrete global attractor for finite difference approximations to the Sine–Gorden and 2D Navier–Stokes equation. In chapters 4,5 and 6 we consider in detail finite difference approximations to the Ginzburg–Landau equation.

It is natural to ask what is the relationship between the discrete and continuous global attractors. We may seek to prove two types of results: upper–semicontinuity and lower–semicontinuity of the discrete attractor to the true attractor. Roughly speaking upper–semicontinuity yields that in the limit the approximate global attractor is contained in the true attractor whereas lower–semicontinuity yields that in the limit the true attractor is contained in the approximate global attractor. Together these results yield set convergence in the Hausdorff metric.

Upper–semicontinuity is the easier result to prove, but even so this is non–trivial for many problems. In the infinite dimensional case Hale et al [64] consider upper–semicontinuity of the global attractors of finite element and spectral approximations to the global attractors of reaction diffusion equations. For convection–diffusion equations upper–semicontinuity of linear mutistep methods was considered in [71]. In finite dimensions upper–semicontinuity is considered in [77] for Runge–Kutta methods and for linear multistep methods in [72].

Regularity of solutions is often an issue in proving upper–semicontinuity for partial differential equations, this is discussed for example in Larsson [89] for finite element approximations to reaction–diffusion equations: the results of [89] are extended in [45] to the Cahn–Hilliard equation. Yin Yan presents in [137] an upper–semicontinuity result for finite difference approximations to the 2D Navier–Stokes equations.

Lower–semicontinuity is a far more difficult thing to show and in general will not be true for every upper–semicontinuous approximation. Lower–semicontinuity of approximations has only been shown in general when the attractors have a certain structure. For details we refer the reader to [65, 119, 72, 78, 79, 80, 125].

Many of the continuous dynamical systems considered above possess an *inertial manifold.* Following the philosophy of seeking numerical approximations with the same qualitative features as the dynamical systems they approximate: we can ask that the approximations have discrete inertial manifolds. This question is considered in a more general context in [47, 33] or [83] for a class dissipative equations and finite element and spectral approximations.

One may use the existence of an inertial manifold to try and improve numerical schemes. These schemes are generally known as non–linear Galerkin methods and they compute approximate inertial manifolds. For full details see in particular the works of Marion and Temam (for example [94]) and in the context of the Ginzburg–Landau equation Promislow [105]. Recently however the effectiveness of these schemes has been questioned in certain cases: see for example Heywood and Rannacher [70] or Jones et al [82].

In the past decade there has been much work on the approximation of dynamical systems by numerical schemes with qualitatively the same dynamic properties. For a review of the work in finite dimension we refer to Stuart [125] and for the special case of Hamiltonian systems to the review by Sanz–Serna [117]. For the infinite dimensional case there are too many contributions to list individuals here.

The dynamics on an attractor may be very complicated and in some cases the attractor is said to be a *strange* or *chaotic* attractor. The quality that defines the "strangeness" of an attractor was a matter of debate for some years – see for example the article by Kirchgraber and Stoffer [86] for a discussion on different definitions of chaos for discrete dynamical systems. The definition we give of strange or chaotic attractor uses the most common definition found in the context of dynamical systems: sensitivity to initial conditions (see for example Devaney [34]). There are few proofs in the current literature of the existence of chaos in a system and most of these are for discrete dynamical systems such as Smale's horseshoe map which be found in the standard texts such as [34] or [133]. Proofs of chaos in continuous dynamical systems are rarer - however recently Mischaikow [98] has proved the existence of chaos in the Lorenz equations for particular parameter values.

Chaos is often characterised by the existence of positive Lyapunov exponents which we define in section 1.2.3. These exponents were proposed by Lyapunov [93] and are a generalization of linear stability analysis of equilibrium solutions to time dependent solutions. A classical treatment may be found in Sansone and Conti [116] and a more measure theoretic approach in the work of Ruelle [112, 113, 114]. Associated with

Lyapunov exponents is the Lyapunov dimension. It was proposed by Kaplan and Yorke that the Lyapunov dimension of an attractor bounds above the Hausdorff dimension. This is discussed in many papers see for example Constantin and Foias [27] for the Navier–Stokes equation and the paper by Ghidaglia and Héron [55] which uses the same techniques but for the Ginzburg–Landau equation. Although Lyapunov exponents are often estimated analytically such as in [27] or estimated from experimental data such as in [135] they are more commonly estimated numerically, see for example [18, 84] or [134]. In the final section of this chapter we review the standard method proposed by Benettin et al [11, 12] for the estimation of Lyapunov exponents and a differential version of this standard method which was proposed by Goldhirsch et al [57].

Until the work of Dieci et al [36] there had been little or no numerical analysis of the schemes employed for estimating the Lyapunov exponents. We note that their analysis is restricted to the specific case when all exponents are calculated whereas in most applications only the non-negative Lyapunov exponents are of interest and are calculated. In Chapter 2 we introduce new numerical schemes for estimating the Lyapunov exponents and examine the convergence of these schemes using the results of Keller, presented by Sanz–Serna in [117]. We note that our analysis is not restricted to the case when all exponents are calculated. These schemes are then generalized to a wider class of problems such as those considered by Dieci et al in [35]. Chapter 2 concludes with a presentation of numerical results for the Lorenz equations and a comparison of our results with those of other authors.

The remainder of the thesis is principally concerned with the approximation of a particular partial differential equation: the complex Ginzburg–Landau equation.

We start Chapter 3 by introducing the complex Ginzburg–Landau equation and the mathematical framework for the equation. The complex Ginzburg Landau equation models the evolution of the amplitude of perturbations to steady state solutions at the onset of instability. Since it models such a fundamental phenomena the Ginzburg–Landau equation arises in many areas of physics. In fluid dynamics it is sometimes referred to as the Stuart–Stewartson equation and is found, for example, in the study of Poiseuille flow, Rayleigh–Bénard convection and Taylor–Couette flow [123, 84, 41, 110, 111, 39]. The equation is also used to model the transition to turbulence in chemical mediums [75, 76]. The equation derives the name used here from the study of superconductivity where it models the phase transition of the material from superconducting phase to a non–superconducting phase [84, 41, 110, 111, 39]. As a phase transition equation it is closely related to other phase transition equations such as the Cahn-Allen or Chafee–Infante equation (see [2] and [23]) or the Cahn–Hilliard equation [21].

In section 3.2 we consider an abstract evolution equation in a Hilbert space and recall definitions and theorems from the works of Henry [68] and Pazy [104] which allow us to define arbitrary powers of a linear operator A and the smoothing property of a linear semi–group generated by a linear operator. We use the definition of powers of a linear operator to define in section 3.3 the Sobolev spaces (see for example Adams [1]) and the space of Gevrey class of regularity. The definition of Gevrey class and regularity which we give here is the same as that used by [49, 40] and [42] but it is defined in a more general setting in [32, 54, 106].

In Section 3.4 we start our review of dynamical systems results for the complex Ginzburg–Landau equation. To begin we consider the existence of a global attractor, the first proof of which for the Ginzburg-Landau equation was furnished by Ghidaglia and Héron [55]. We sketch the proof of the existence of a global attractor given in [128] which holds in 2 dimensions and then present two 1 dimensional alternatives: the first due to Doering et al [41], the second suggested by Süli [126]. We then discuss the regularity of solutions to the Ginzburg–Landau equation. Bartuccelli et al [9] have examined this question by proving a sequence of estimates in Sobolev spaces to show that solutions are C^∞. A stronger result has been proved by Doelman and Titi [40] for the Ginzburg–Landau equation with cubic non–linearity and by Duan et al [42] for higher order non–linear terms: the solutions lie in a Gevrey class of regularity.

In section 3.5 we continue our revision of results for the Ginzburg–Landau equation and consider the dimensionality of the dynamics. For many equations such as the Kuramoto–Sivashinsky equation, the existence of an inertial manifold has been proved, see for example the works of Constantin et al [29] Foias et al [48], Foias and Titi [50]. For the complex Ginzburg–Landau equation the existence of an inertial manifold was first proved by Doering et al [41]. In section 3.5.1 we outline a proof based on the work in [83] and [48] of an inertial manifold and in section 3.5.2 we outline a proof of the cone condition by [41].

Ghidaglia and Héron [55] and Doering et al [41] obtain upper bounds on the Lyapunov dimension of the attractor and hence via the Kaplan–Yorke conjecture on the Hausdorff dimension. This work is summarized in section 3.5.3. More recently Kukavica [87] has proved that solutions to the Ginzburg–Landau equation are completely determined by the values at two points. He uses this remarkable result to bound the fractal set of stationary solutions. Numerically the finite dimensionality of the Ginzburg–Landau equation has been investigated by Keefe [84] and Rodriguez et al [110, 111] among others. We conclude our revision of work on the Ginzburg–Landau equation by re–examining some of the analysis in [41] on exact solutions and their stability.

In section 3.6 we introduce the finite difference discretizations of the complex Ginzburg–Landau equation that we consider for the remainder of the thesis. In the final section of Chapter 3 we introduce the mathematical framework for the discrete equations by defining relevant norms on the vector space $\mathbb{C}^J$ and prove discrete versions of various Sobolev space inequalities. Although some results may be found in the literature – for example in Mokin [100], the works of Yan [136, 137, 138] or Yulin [142] among others – the proofs we present in section 3.7 are our own and were found independently.

We start Chapter 4 by considering the semi–discrete approximation to the Ginzburg–Landau equation i.e. a purely spatial discretization. We prove that the resulting set of ordinary differential equations forms a dynamical system and that there exists absorbing balls in discrete L^2 and H^1 spaces of radius independent of initial data and the spatial mesh size Δx. We conclude the existence of a global attractor. In section 4.2 we prove that solutions to the semi–discrete approximations lie in a discrete Gevrey class of regularity: the method of proof follows that of [40] and [42]. We use this result to prove upper–semicontinuity of the semi–discrete global attractor to the global attractor of the Ginzburg–Landau equation. For finite difference approximations the only other results proving upper–semicontinuity that we are aware of are due to Yan in [137]. Our method differs to that of Yan who uses a piecewise linear interpolation and sets the analysis in the space L^2.

In section 4.4 we consider fully discrete approximations to the Ginzburg–Landau equation. For a fully implicit scheme we prove the existence of absorbing balls in the discrete spaces L^2 and H^1 of radius independent of the initial data and the mesh sizes. Existence of a global attractor is also shown. These results hold provided the spatial and temporal mesh sizes are taken sufficiently small – as some function of the parameters. Although this scheme has good dynamic qualities, the implementation of this scheme requires a non–linear solver. In the following section we examine a fully discrete scheme which only requires a tri–diagonal solver and hence is much cheaper to solve. We prove the existence of absorbing balls in the discrete L^2 independent of mesh sizes and the initial condition. Again we prove the existence of a global attractor. In this case we find that the meshes must be taken sufficiently small and that the temporal mesh is now a function of the spatial mesh. From numerical experiments it appears that this dependence is a product of the analysis. Chapter 4 concludes with a presentation and discussion of numerical results on four finite difference schemes for the complex Ginzburg–Landau equation.

In Chapter 5 we prove that, provided the spatial resolution is sufficiently small, the semi–discrete and fully implicit complex Ginzburg–Landau equation admits an inertial manifold. The proof is based on the proof in [47] or [83]. Although not required for the proof of an inertial manifold we show that both these approximations satisfy the cone condition given in [41] for the continuous equation. Similar results may be found in [50] for the Kuramoto–Sivashinsky equation. For the semi–discrete approximation we also obtain analytically bounds on the sum of Lyapunov exponents leading to bounds on the Lyapunov dimension and via the Kaplan–Yorke conjecture to bound the Hausdorff dimension - see the work of Constantin and Foias [27]. The bounds we obtain agree with the analysis in [41] for the continuous case. Similar analysis is performed for the semi–discrete approximation to the 2D Navier–Stokes equations in [136].

Chapter 6 examines the existence and stability of discrete forms of exact solutions for three finite difference approximations to the Ginzburg–Landau equation. This is similar to the analysis in [41] for the continuous case. Numerical results are presented for the schemes.

We change tack in Chapter 7 and examine spectral approximations to the Ginzburg–

Landau equation with both periodic and Dirichlet boundary conditions. Using the numerical continuation code AUTO [37] we re–examine the bifurcation structure numerically. For the Dirichlet case Mischaikow and Morita [97] have shown that for a particular range of parameters the Ginzburg–Landau equations may be transformed to the Chafee-Infante equation [2, 23]. There is a large literature on the Chafee–Infante, which is a gradient system: see for example [68] and for the coupled system [7, 6, 5]. Computational work on the structure of the attractor may be found in Bai et al[3] using the methods of Beyn [13] and [38], to compute heteroclinic orbits. Also considered in [3] is the direct approximation of the global attractor of the Cahn–Hilliard equation and the relation between the global attractors for the Cahn-Hilliard and Chaffee–Infante. Another way the attractor may be approximated directly is the numerical computation of homoclinic connections, see for example [14, 99] or [24]. Finally we present some preliminary results towards the computation of heteroclinic orbits using the methods of [13, 38] and Bai et al [4] for the complex Ginzburg–Landau equation.

1.2 Dynamical Systems

As advertised we start by giving some standard dynamical systems definitions and results. These may be found in most of the standard text books such as [26, 34, 59, 68, 74, 104, 128] and [133] or in review papers such as [44] or [125]. In this section we attempt to treat continuous and discrete systems simultaneously. We differentiate between them by flagging with a **C** for continuous and **D** for discrete.

Let X be a Banach space with norm $\|\bullet\|$ and identity operator $I : X \to X$. We start by making some standard definitions on the space X.

Definition 1.2.1 (Semi–Distance) Let $A, B \subset X$ and let $u \in X$, then

- the *distance of a point to a set* is defined by

$$\mathrm{dist}_X(u, A) := \inf_{v \in A} \|u - v\|;$$

- and the *distance between two sets* A, B is defined by

$$\mathrm{dist}_X(B, A) := \sup_{u \in B} \mathrm{dist}_X(u, A).$$

Definition 1.2.2 (Hausdorff Distance) The *Hausdorff distance* $d_H(A, B)$ between two sets $A, B \subset X$ is defined by

$$d_H(A, B) := \max\left\{dist_X(A, B), dist_X(B, A)\right\}.$$

Definition 1.2.3 (ϵ–Neighbourhood)

We define the ϵ*–neighbourhood* of a set $A \subset X$ by

$$N(A, \epsilon) := \left\{u \in X : dist_X(u, A) < \epsilon\right\}.$$

We consider in this chapter the following problems:

C Given $U(0) = U^0 \in X$, find $U(t) \in C(\mathbb{R}^+, X)$ which satisfies

$$\frac{d}{dt}U(t) = f(U(t)). \tag{1.2.1}$$

D Given $U^0 \in X$, find $U^{n+1} \in X$ which satisfies

$$U^{n+1} = g(U^n). \tag{1.2.2}$$

Examples These will be referred to throughout this chapter.

Continuous Case

c1 Let $X = \mathbb{R}^p$ and consider

$$\frac{d}{dt}U(t) = f(U(t)), \quad U^0 \in \mathbb{R}^p,$$

where $f : \mathbb{R}^p \to \mathbb{R}^p$ is given by

$$f(v) = Av$$

and A is a $p \times p$ matrix with constant entries.

c2 The Lorenz Equations (Lorenz 1963 [92]).

Let $X = \mathbb{R}^3$ and consider

$$\begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix} = \begin{pmatrix} \sigma(y - x) \\ rx - y - xz \\ xy - bz \end{pmatrix} \tag{1.2.3}$$

where the parameters $b, r, \sigma > 0$. In the notation of (1.2.1) we have $U = (x, y, z)^T$ and $f : \mathbb{R}^3 \to \mathbb{R}^3$ is given by the right hand side of (1.2.3). Further details on the Lorenz equations may be found in Sparrow [122].

Discrete Case

d1 Let $X = \mathbb{R}^p$ and consider the explicit Euler approximation to **c1**

$$\frac{U^{n+1} - U^n}{\Delta t} = AU^n$$

where $U^0 \in \mathbb{R}^p$ is given , A is the $p \times p$ matrix with constant entries as in **c1** and Δt is a parameter in $\mathbb{R}^+$. Note that this may be re–written as

$$U^{n+1} = g(U^n) := (I + \Delta t A)\, U^n.$$

d2 Let $X = \mathbb{R}^p$ and consider the implicit Euler approximation to **c1** given by

$$\frac{U^{n+1} - U^n}{\Delta t} = AU^{n+1}$$

where $U^0 \in \mathbb{R}^p$ is given , A is the $p \times p$ matrix with constant entries as in **c1** and Δt is a parameter in $\mathbb{R}^+$. Note that this may be written as

$$U^{n+1} = g(U^n) := (I - \Delta t A)^{-1} U^n,$$

provided the matrix $I - \Delta t A$ is non–singular.

We now define mathematically what we mean by a dynamical system.

Definition 1.2.4

C The continuous system (1.2.1) is said to define a *local dynamical system* on an open set $E \subseteq X$ if there exists a time $t^*(U)$ such that for every U^0 in E there exists a unique solution $U(t)$ of (1.2.1) remaining in E defined for all t s.t. $0 \leq t < t^*(U)$.

If $t^*(E) = \infty$ then we say that (1.2.1) defines a *dynamical system* on E.

D The discrete system (1.2.2) is said to define a *local dynamical system* on an open set $E \subseteq X$ if there exists a time $n^*(U)$ such that for every U^0 in E there exists a unique solution $(U^n)_{n=0}^{n^*-1}$ of (1.2.1) remaining in E.

If $n^*(E) = \infty$ then we say that (1.2.2) defines a *dynamical system* on E.

The mappings that we shall consider most often in this thesis are approximations to continuous dynamical systems - just as examples **d1** and **d2** are approximations to the continuous system **c1**.

1.2.1 Semi–Group Formulation and Some Useful Definitions

We shall use the semi–group formulation to describe dynamical systems. Further details on semi–group theory and its application to dynamical systems may be found for example in Henry [68], Pazy [104] or Temam [128].

Definition 1.2.5

C Given a continuous dynamical system we define the continuous operator $S(t) : E \to E$, by

$$U(t) = S(t)U^0 \quad \forall t \geq 0$$

The set $\{S(t)\}_{t\geq 0}$ enjoys the usual semi–group properties i.e.

$$S(t_1 + t_2) = S(t_1) \cdot S(t_2), \quad \text{and} \quad S(0) = I,$$

and hence is termed a *semi–group of continuous operators.*

D In the discrete case we define the continuous operator $S^n : E \to E$ by

$$U^n = S^n U^0 \quad \forall n \in \mathbb{N}.$$

The set $\{S^n\}_{n\in\mathbb{N}}$ enjoys the usual semi–group properties i.e.

$$S^{n+m} = S^n \cdot S^m \quad \text{and} \quad S^0 = I$$

and hence is termed a *discrete semi–group of continuous operators.*

We now define the action of a semi–group on a set.

Definition 1.2.6 *Let $E \subseteq X$, then we define the the semi–group acting on E by*

$$\textbf{C} \quad S(t)E = \bigcup_{x\in E} S(t)x; \qquad \textbf{D} \quad S^n E = \bigcup_{x\in E} S^n x.$$

Examples

c1 We may solve **c1** to find

$$U(t) = e^{tA}U^0$$

and since e^{tA} is a continuous operator we have a continuous dynamical system with $S(t) = e^{tA}$.

d1 We have that $U^{n+1} = (I + \Delta t A)^n U^0$ and thus since $I + \Delta t A$ is continuous we have a continuous dynamical system with $S^n := (I + \Delta t A)^n$.

The following definitions will be particularly useful when discussing the long–time evolution of dynamical systems.

Definition 1.2.7

Given $U^0 \in X$ we define the *orbit* or *trajectory* starting at U^0 to be the set

$$\mathbf{C} \quad \bigcup_{t \geq 0} S(t)U^0; \qquad\qquad \mathbf{D} \quad \bigcup_{n \in \mathbb{N}} S^n U^0.$$

The long–time behaviour of a point or set is given by the ω–limit set.

Definition 1.2.8 Given $U^0 \in X$, the *ω-limit set* of a point U^0 is defined as

$$\mathbf{C} \quad \omega(U^0) := \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} S(t)U^0}; \qquad \mathbf{D} \quad \omega(U^0) := \bigcap_{m \in \mathbb{N}} \overline{\bigcup_{n \geq m} S^n U^0}.$$

The ω–limit set of $A \subset X$ is defined by

$$\mathbf{C} \quad \omega(A) := \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} S(t)A}; \qquad \mathbf{D} \quad \omega(A) := \bigcap_{m \in \mathbb{N}} \overline{\bigcup_{n \geq m} S^n A}.$$

Definition 1.2.9 A set $A \subset X$ is *positively invariant* for the semi–group $S(t) : X \to X$ $(S^n : X \to X)$ if

$$\mathbf{C} \quad S(t)A \subset A, \quad \forall t \geq 0; \qquad \mathbf{D} \quad S^n A \subset A, \quad \forall n \in \mathbb{N}.$$

A set $A \subset X$ is said to be invariant if

$$\mathbf{C} \quad S(t)A = A, \quad \forall t \geq 0; \qquad \mathbf{D} \quad S^n A = A, \quad \forall n \in \mathbb{N}.$$

Note Let $A \subset X$ be invariant for $S(t) : X \to X$ (or $S^1 : X \to X$). Then $S(t) : A \to A$ (or $S^1 : A \to A$) is injective and so we may define the inverse $S(-t)$ (or S^{-1}).

The ω–limit set gives the long forward behaviour of a point, for completeness we now define the long–time backwards evolution of a point and set.

Definition 1.2.10 Given $U^0 \in X$, then when it exists, the *α-limit set* is defined as

$$\mathbf{C} \quad \alpha(U^0) := \bigcap_{s \leq 0} \overline{\bigcup_{t \leq s} S(-t)^{-1}U^0}; \qquad \mathbf{D} \quad \alpha(U^0) := \bigcap_{m \in \mathbb{N}} \overline{\bigcup_{n \geq m} S^{-n} U^0}.$$

The α–limit set of $A \subset X$ when it exists is defined by

$$\mathbf{C} \quad \alpha(A) := \bigcap_{s \leq 0} \overline{\bigcup_{t \leq s} S(-t)^{-1}A}; \qquad \mathbf{D} \quad \alpha(A) := \bigcap_{m \in \mathbb{N}} \overline{\bigcup_{n \geq m} S^{-n} A}.$$

Possibly the simplest example of an invariant set is an *equilibrium, stationary* or *fixed point/solution.*

Definition 1.2.11 A *stationary or equilibrium solution* is a solution $\widetilde{U}$ of either **C** (1.2.1) or **D** (1.2.2) such that

$$\mathbf{C} \qquad S(t)\widetilde{U} = \widetilde{U} \ \ \forall t \geq 0 \ ; \qquad \mathbf{D} \qquad S^n\widetilde{U} = \widetilde{U} \ \forall n \in \mathbb{N}.$$

Example

c2 The Lorenz equations have either one or three stationary points depending on the parameter r. Setting the right hand side of the Lorenz equations to zero we find either

$$x = y = z = 0$$

or

$$x = y = \pm\sqrt{b(r-1)} \quad \text{and} \quad z = r - 1.$$

Thus the origin is a stationary solution for all parameter values whereas the other two points are stationary $\forall r \geq 1$.

Definition 1.2.12

C Let $\widetilde{U}$ be an equilibrium solution to (1.2.1) then the *local stable manifold* $W^s_{loc}(\widetilde{U})$ and *stable manifold* $W^s(\widetilde{U})$ are defined by:

$$\begin{aligned} W^s_{loc}(\widetilde{U}) &:= \left\{U^0 \in X : U(t) \in N(\widetilde{U}, \epsilon) \ \forall t \geq 0 \ \& \lim_{t\to\infty} U(t) = \widetilde{U}\right\}; \\ W^s(\widetilde{U}) &:= \left\{U^0 \in X : \lim_{t\to\infty} U(t) = \widetilde{U}\ \right\}. \end{aligned}$$

When they exist the *local unstable manifold* $W^u_{loc}(\widetilde{U})$ and *unstable manifold* $W^u(\widetilde{U})$ of the equilibrium solution $\widetilde{U}$ are defined by:

$$\begin{aligned} W^u_{loc}(\widetilde{U}) &:= \left\{U \in X : U(t) \in N(\widetilde{U}, \epsilon) \ \forall t \ \leq 0 \ \& \lim_{t\to-\infty} U(t) = \widetilde{U}\right\}. \\ W^u(\widetilde{U}) &:= \left\{U \in X : \lim_{t\to-\infty} U(t) = \widetilde{U}\right\}. \end{aligned}$$

D The *local stable manifold*, $W^s_{loc}(\widetilde{U})$ and *stable manifold*, $W^s(\widetilde{U})$, of the equilibrium solution $\widetilde{U}$ are defined by:

$$\begin{aligned} W^s_{loc}(\widetilde{U}) &:= \left\{U \in X : U^n \in N(\widetilde{U}, \epsilon) \forall n \in \mathbb{N} \ \& \lim_{n\to\infty} U^n = \widetilde{U}\right\}; \\ W^s(\widetilde{U}) &:= \left\{U \in X : \lim_{n\to\infty} U^n = \widetilde{U}\right\}. \end{aligned}$$

When they exist the *local unstable manifold* $W^u_{loc}(\widetilde{U})$ and *unstable manifold*, $W^u(\widetilde{U})$, of the equilibrium solution $\widetilde{U}$ are defined by:

$$\begin{aligned} W^u_{loc}(\widetilde{U}) &:= \left\{U^0 \in X : U^{-n} \in N(\widetilde{U},\epsilon)\ \forall n \in \mathbb{N}\ \&\ \lim_{n\to\infty} U^{-n} = \widetilde{U}\right\}. \\ W^u(\widetilde{U}) &:= \left\{U \in X : \lim_{n\to-\infty} U^{-n} = \widetilde{U}\ \forall n \in \mathbb{N}\right\}. \end{aligned}$$

Often we are not only interested in equilibrium solutions but in orbits which connect equilibriums.

Definition 1.2.13 Let $\widetilde{U}$ and $\widetilde{V}$ be two equilibrium solutions of either (1.2.1) or (1.2.2). Then $\widetilde{U}$ is said to be *connected* to $\widetilde{V}$ if there exists an orbit γ such that

$$\alpha(\gamma) = \widetilde{U} \ \text{ and } \ \omega(\gamma) = \widetilde{V}.$$

When $\widetilde{U} = \widetilde{V}$ then this is termed a *homoclinic connection*, and when $\widetilde{U} \neq \widetilde{V}$ this is termed a *heteroclinic connection.* When the global stable and unstable manifolds of $\widetilde{U}$ and $\widetilde{V}$ exist then heteroclinic connections exist provided

$$W^u(\widetilde{U}) \bigcap W^s(\widetilde{V}) \neq \phi.$$

We now state mathematically what we mean by attracting and attractor.

Definition 1.2.14 A set $\mathcal{A} \subset X$ is said to *attract* a set $B \subset X$ if for any $\epsilon > 0$ $\exists\ t_0 = t_0(\epsilon, \mathcal{A}, B)$ $(n_0 = n_0(\epsilon, \mathcal{A}, B))$ such that

$$\textbf{C}\quad S(t)B \subset N(\mathcal{A},\epsilon) \quad \forall t \geq t_0,$$

$$\textbf{D}\quad S^n B \subset N(\mathcal{A},\epsilon) \quad \forall n \geq n_0.$$

Definition 1.2.15 A set $\mathcal{A} \subset X$ is said to be an attractor if $\mathcal{A}$ is a compact, invariant set which attracts an open neighbourhood of itself.

Definition 1.2.16 The *basin of attraction* of an attractor $\mathcal{A} \subset X$ is defined to be the set

$$\{U^0 \in X : \omega(U^0) \in \mathcal{A}\}.$$

As defined in 1.2.15 an attractor is a local object, attracting only a subset of X. Our next definition defines a global attracting object, one which attracts all initial conditions.

Definition 1.2.17

We say that $\mathcal{A} \subset X$ is a *global* or *universal attractor* for the semi–group $\{S(t)\}_{t\geq 0}$ (or $\{S^n\}_{n\in\mathbb{N}}$) if $\mathcal{A}$ is a compact attractor that attracts all the bounded sets of X.

Note that convergence to the attractor may be arbitrarily slow, thus we are also interested in finding sets into which bounded sets enter after a finite time, these are termed "absorbing sets" and are defined below.

Definition 1.2.18

A closed bounded subset $\mathcal{B}$ of E an open set in X is said to be *absorbing* in E if for each bounded set $B \subset E$ there exists $t_0(B) \geq 0$ ($n_0(B) \in \mathbb{N}$) such that

$$\mathbf{C} \quad S(t)B \subset \mathcal{B}\ \forall\ t \geq t_0; \qquad \mathbf{D} \quad S^n B \subset \mathcal{B}\ \forall\ n \geq n_0.$$

The following theorem gives us a tool for proving the existence of a global attractor; first however we require another definition.

Definition 1.2.19 An operator $S(t) : X \to X$ ($S^n : X \to X$) is said to be uniformly compact if for every bounded set $\mathcal{B} \subset X$ there exists $t_0 = t_0(\mathcal{B}) > 0$ ($n_0 = n_0(\mathcal{B}) \in \mathbb{N}$) such that $\bigcup_{t\geq t_0} S(t)\mathcal{B}$ ($\bigcup_{n\geq n_0} S^n\mathcal{B}$) is relatively compact in X.
Recall that a set is relatively compact if its closure is compact.

Theorem 1.2.1

Assume there exists $t_0 > 0$ ($n_0 \in \mathbb{N}$) such that for all $t \geq t_0$ ($n \geq n_0$) such that the operator $S(t) : X \to X$ ($S^n : X \to X$) is uniformly compact. Further suppose there exists open set $E \subset X$, bounded set $\mathcal{B} \subset E$ such that $\mathcal{B}$ is an absorbing set in E.

Then $\mathcal{A}$ defined by

$$\mathcal{A} := \omega(\mathcal{B})$$

is a global attractor.

Furthermore if E is convex and connected then $\mathcal{A}$ is connected too.

Proof See Hale [63] and Temam [128].□

Example c2 The Lorenz Equations.

We start by making a change of variables. Let $z = z - r - \sigma$. Then,

$$\begin{aligned} x_t &= \sigma(y-x) \\ y_t &= -(\sigma x + y + xy) \\ z_t &= -b(r+\sigma) + xy - bz \end{aligned}$$

With $U = (x,y,z)^T$, take the inner–product in $\mathbb{R}^3$ with the above

$$\frac{1}{2}\frac{d}{dt}|U|^2 = -\sigma x^2 - y^2 - bz^2 - b(r+\sigma)z.$$

So,

$$\frac{1}{2}\frac{d}{dt}|U|^2 + \sigma x^2 + y^2 + bz^2 \le (b-1)z^2 + \frac{b^2}{4(b-1)}(r+\sigma)^2$$

Now let $\delta = \min(1,\sigma)$ to get

$$\frac{d}{dt}|U|^2 + 2\delta|U|^2 \le \frac{b^2}{(b-1)}(r+\sigma)^2.$$

Hence,

$$|U(t)|^2 \le |U(0)|^2 e^{-2\delta t} + \frac{b^2}{4\delta(b-1)}(r+\sigma)^2(1-e^{2\delta t}). \tag{1.2.4}$$

From (1.2.4) it follows that

$$\limsup_{t\to+\infty} |U(t)| \le \rho_0$$

where,

$$\rho_0^2 = \frac{b^2}{4\delta(b-1)}.$$

Therefore any ball $\mathcal{B}(0,\rho)$ centre 0, radius $\rho > \rho_0$ is an *absorbing set.* Given any bounded set $\mathcal{B}_0$ of $H = \mathbb{R}^3$, included in a ball $B(0,R)$, then $S(t)\mathcal{B}_0 \subset \mathcal{B}(0,\rho)$ for $t \ge t(\mathcal{B}_0)$,

$$t(\mathcal{B}_0) = \frac{1}{2\delta}\log\frac{R^2}{\rho^2-\rho_0^2}.$$

The existence of a *global attractor* follows from inequality (1.2.4) and Theorem 1.2.1.

In general the existence of the global attractor does not tell us about the structure of the attractor. The Lorenz equations are infamous for their "chaotic attractor", however the quality which makes them chaotic was a matter of debate for several years. The most widely accepted defining quality of chaos in the field of dynamical systems is sensitivity to initial conditions.

Definition 1.2.20 (Sensitivity to Initial Conditions)

The semi–group $S(t) : X \to X$ ($S^n : X \to X$) is said to have *sensitive dependence to initial conditions* if there exists $\delta > 0$ such that for any $u \in X$ and any open neighbourhood E of u there exists $v \in E$ and $T > 0$ ($N \in \mathbb{N}$) such that

$$\mathbf{C} \quad \|S(T)u - S(T)v\| > \delta; \qquad \mathbf{D} \quad \|S^N u - S^N v\| > \delta.$$

Sensitivity to initial conditions ensures that two initially close conditions will separate, which gives unpredictability to the system. Another important quality of chaotic systems or attractors is irreducibility or topological transitivity.

Definition 1.2.21 (Topological Transitive)

The semi–group $S(t) : X \to X$ ($S^n : X \to X$) is said to *topologically transitive* on $E \subset X$ if for any pair of open sets $A, B \subset E$ there exists $T > 0$ ($N \in \mathbb{N}$) such that $(S(t)A) \bigcap B \neq 0$ ($(S^n A) \bigcap B \neq 0$).

Thus we make the following definition of chaos, see for example [133].

Definition 1.2.22 An attractor $\mathcal{A}$ is said to be *chaotic* or a *strange attractor* if $S(t) : X \to X$ ($S^n : X \to X$)

- has sensitive dependence to initial conditions;
- and is topologically transitive on $\mathcal{A}$.

We note that some authors, such as Devaney [34] include a third criteria for a chaotic attractor, namely that periodic orbits be dense on $\mathcal{A}$. This gives an element of regularity to the attractor in addition to the random behaviour.

It has been proved by Mischaikow [96], that for a certain parameter regime the Lorenz equations are indeed chaotic. For a wide range of other parameter values there is excellent numerical evidence that the Lorenz equations exhibit chaos. In section 1.2.3 we give a characterization of chaos, as well as defining a useful tool for determining the asymptotic state of the dynamical system, and numerically detecting chaos.

1.2.2 Stationary Solutions and Stability

Given $U^0 \in X$ suppose we have found a solution $S(t)U^0$ to (1.2.1) or S^nU^0 to (1.2.2), then it is natural to ask whether the solution is attracting or not. To determine this we examine the evolution of an ϵ-neighbourhood about the solution. In this section we outline the analysis of Hirsch and Smale [74] and Guckenheimer and Holmes [59] and shall treat the continuous and discrete systems separately.

Definition 1.2.23 (Lyapunov Stability)

Let $U(t)$, $t \geq 0$ (or U^n, $n \in \mathbb{N}$) be any solution to (1.2.1) (or (1.2.2)). Then $U(t)$ is said to be *Lyapunov stable* if, given $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that for any other solution $V(t)$ (or V^n) satisfying $||U^0 - V^0|| < \delta$, then

$$\mathbf{C} \quad ||U(t) - V(t)|| < \epsilon \;\; \forall t \geq 0; \qquad \mathbf{D} \quad ||U^n - V^n|| < \epsilon \;\; \forall n \in \mathbb{N}.$$

Definition 1.2.24 (Asymptotically Stability) *Let $U(t)$, $t \geq 0$ (or U^n, $n \in N$) be any solution to* (1.2.1) *(or* (1.2.2)*) and let $\widetilde{U}$ be an equilibrium solution. Then $\widetilde{U}$ is said to be* asymptotically stable *if it is Lyapunov stable and $\exists b > 0$ such that*

$$\mathbf{C} \;\; ||U^0 - V^0|| < b \implies \lim_{t \to \infty} ||U(t) - V(t)|| = 0;$$

$$\mathbf{D} \;\; ||U^0 - V^0|| < b \implies \lim_{n \to \infty} ||U^n - V^n|| = 0.$$

Note A solution which is not stable is said to be *unstable.*

We now present a method for examining the stability of equilibrium solutions. Since the analysis differs slightly for the continuous and discrete systems we consider the two cases separately.

Continuous Case

Linear Stability Analysis

Given $U^0 \in X$ let $U(t)$ be the solution of the differential equation (1.2.1) at time t. To determine the stability of the solution we consider the *linear evolution equation* or *stability equation* given by

$$\xi_t = Df[U(t)]\xi(t) \tag{1.2.5}$$

where $Df[U(t)]$ denotes the Fréchet derivative of f with respect to U evaluated at $U(t)$.

The stability equation (1.2.5) can be solved to get

$$\xi(t) = L(t, U^0)\xi(0) \tag{1.2.6}$$

where $L(t, U^0)$ is the linearised flow

$$L(t, U^0) = DS[U(t)] \tag{1.2.7}$$

and $DS[U(t)]$ is the linearization with respect to $U(t)$ of the non–linear flow $S(t)$.

The *stability equation* describes the evolution of linear perturbations ξ to the solution $U(t)$ and the growth or decay of the perturbations ξ depends on the eigenvalues of $Df[U(t)]$. If the perturbations decay then the solution is asymptotically stable.

Example : Let $X = \mathbb{R}^p$, then (1.2.6) becomes

$$\xi(t) = L(t, U^0)\xi(0) = e^{tDf[U(t)]}\xi(0). \tag{1.2.8}$$

For the special case when $U(t)$ is a stationary solution $\widetilde{U}$, and hence independent of t, equation (1.2.8) becomes

$$\xi(t) = e^{tDf[\widetilde{U}]}\xi(0).$$

We let $\{\lambda_i\}_{i=1}^{p}$ denote the eigenvalues of the Jacobian matrix $Df[\widetilde{U}]$ and let

- $V_1, \cdots, V_s$ be the (generalised) eigenvectors of $Df[\widetilde{U}]$ whose eigenvalues have negative real part;
- $V_{s+1}, \cdots, V_u$ be the (generalised) eigenvectors of $Df[\widetilde{U}]$ whose eigenvalues have positive real part;
- $V_{u+1}, \cdots, V_p$ be the (generalised) eigenvectors of $Df[\widetilde{U}]$ whose eigenvalues have zero real part.

Then we can define the following linear analogies of the local stable and unstable manifolds defined in Definition 1.2.12.

- Stable Subspace $E^s := \text{span}\{V_1, \cdots, V_s\}$
- Unstable Subspace $E^u := \text{span}\{V_{s+1}, \cdots, V_u\}$
- Centre Subspace $E^c := \text{span}\{V_{u+1}, \cdots, V_p\}$.

For the linear system 1.2.5 the stability of the fixed point is determined by the eigenvalues λ_i of $DS[\widetilde{U}]$. The equilibrium $\widetilde{U}$ is stable if

$$\mathrm{Re}\,(\lambda_i) < 0 \ \ \forall i = 1, \cdots, p.$$

Along the subspace E^s we have a flow which contracts exponentially to the fixed point, along the subspace E^u we have exponential flow away from the stationary solution. This leads to the following defintion

Definition 1.2.25 An equilibrium $\widetilde{U}$ for the continuous system (1.2.1) is said to be *hyperbolic* provided no eigenvalue of $Df[\widetilde{U}]$ has zero real part.

Discrete Case

Linear Stability Analysis

Given $U^0 \in X$ let U^n be the solution of (1.2.2). To determine the stability of the solution we consider the *linear evolution equation* or *stability equation* given by

$$\xi^{n+1} = Df[U^n]\xi^n \tag{1.2.9}$$

where $Df[U^n]$ denotes the Fréchet derivative of f evaluated at U^n.

The stability equation 1.2.9 can be solved to get

$$\xi^{n+1} = L(n, U^0)\xi(0) \tag{1.2.10}$$

where $L(n, U^0)$ is the linearised flow given by

$$L(n, U^0) = DS^1[U^n] \tag{1.2.11}$$

The *stability equation* describes the evolution of linear perturbations ξ^n to the solution U^n and the growth or decay of the perturbations ξ^n depends on the eigenvalues of $Df[U^n]$.

Example : Let $X = \mathbb{R}^p$ and consider the special case when U^n is an equilibrium solution $\widetilde{U}$. We linearise about the equilibrium to get

$$\xi^{n+1} = DS^1[\widetilde{U}]\xi^n \tag{1.2.12}$$

where $DS^1[\widetilde{U}]$ is the linearization of the flow S^1 with respect to $\widetilde{U}$. Since $\widetilde{U}$ is a fixed point, this is a constant matrix. We can solve (1.2.12) to get

$$\xi^{n+1} = DS^n[\widetilde{U}]\xi^0, \tag{1.2.13}$$

where $DS^n[\widetilde{U}] = DS[\widetilde{U}] \cdot DS[\widetilde{U}] \cdots DS[\widetilde{U}]$. In an analogous manner to the continuous case we define stable, unstable and centre subspaces for the map.

- Stable Subspace

  $E^s = \mathrm{Sp}\{$ generalised eigenvectors with eigenvalues of modulus $< 1\}$

- Unstable Subspace

  $E^u = \mathrm{Sp}\{$ generalised eigenvectors with eigenvalues of modulus $> 1\}$

- Centre Subspace

  $E^c = \mathrm{Sp}\{$ generalised eigenvectors with eigenvalues of modulus $= 1\}$

Now we find that the fixed point for the linear system in (1.2.12) is stable provided the eigenvalues of the matrix $DS^n[\widetilde{U}]$ all have modulus less than one. The stable subspace gives the direction in which perturbations to the fixed point decay, and the unstable subspace gives the direction in which perturbations grow. This leads to the following definition

Definition 1.2.26 An equilibrium $\widetilde{U}$ for the discrete system (1.2.2) is said to be *hyperbolic* provided no eigenvalue of $Df[\widetilde{U}]$ lies on the unit circle (or equivalently has modulus equal to 1).

In the following section we introduce a generalization of the linear stability analysis for equilibrium solutions.

1.2.3 Lyapunov Exponents

In section 1.2 we defined entities such as attractors and global attractors: these are sets that trajectories approach as $t \to \infty$. As stated before the dynamics on these sets may be very complicated, especially in the presence of chaos. One meaningful way of characterizing the nature of the dynamics on an attractor is to calculate the Lyapunov exponents [93]. Lyapunov exponents measure the average exponential rate of convergence or expansion in the trajectories of arbitrarily close initially conditions. Since this gives a measure of the *sensitivity to initial conditions* that is used to define chaos (Definition 1.2.22) they are often used to characterize chaos, and because of this Lyapunov exponents are considered in a wide range of scientific subjects.

Lyapunov exponents are often estimated analytically see for example [27, 55] and [128, Chapter 5]. More commonly Lyapunov exponents are estimated numerically such as in [12, 18, 36, 57, 84, 120] and [134] or are estimated from experimental data for example as in [44, 101, 115, 132, 135]. Much of the theory behind Lyapunov exponents is beyond the scope of this thesis and we refer the reader in particular to the review Ruelle and Eckmann [44] and the work of Ruelle [112, 113, 114] and the many references contained therein for a more measure–theoretic approach. A more classical treatment of Lyapunov exponents set in $\mathbb{R}^p$ may be found in Sansone and Conti [116] . Although in section 5.2 we estimate Lyapunov exponents analytically, we will principally be concerned with numerical estimation of Lyapunov exponents: the analysis of the schemes (see Chapter 2) and their estimation for the information they yield (see Sections 4.5, 7.3.1 and 7.4.2).

For the purposes of this section let X be a Hilbert space with inner–product $< \bullet, \bullet >$ and induced norm $\| \bullet \|^2 =< \bullet, \bullet >$.

Consider the linear equation in X,

$$\frac{d}{dt}\xi = A(t)\xi(t) \qquad \text{given } \xi(0) = \xi^0 \in X; \tag{1.2.14}$$

and the linear map

$$\xi^{n+1} = A(n)\xi^n \qquad \text{given } \xi^0 \in X. \tag{1.2.15}$$

We solve these linear equations to get

$$\mathbf{C} \quad \xi(t) = L(t)\xi^0 \qquad \mathbf{D} \quad \xi^{n+1} = L(n)\xi^0. \tag{1.2.16}$$

We shall consider the continuous and discrete cases simultaneously by considering the linear operator $L(t)$ with $t \in \mathbb{R}^+$ and $t \in \mathbb{N}$. Further we assume that $L : X \times \mathbb{R} \to X$ is a compact operator and we let $L(t)^*$ denote the adjoint operator. We are now in a position to define Lyapunov exponents.

Definition 1.2.27 Whenever they exist let $\{\xi_i^0\}$ be the set of orthonormal eigenvectors for the operator $(L(t)^*L(t))^{1/2}$ with corresponding eigenvalues $\alpha_i = \alpha_i(L(t))$: so that

$$(L(t)^*L(t))^{1/2}\,\xi_i^0 = \alpha_i \xi_i^0.$$

Then the ith *local Lyapunov exponent* μ_i is defined by

$$\mu_i := \lim_{t\to\infty} \frac{1}{t} \log \alpha_i \tag{1.2.17}$$

whenever the limit exists. Equivalently (see Lemma 1.2.1) we have

$$\mu_i = \lim_{t\to\infty} \frac{1}{t} \log \|L(t)\xi_i^0\|. \tag{1.2.18}$$

We define the ith *approximate Lyapunov exponent* $\mu_i(t)$ by

$$\mu_i(t) := \frac{1}{t} \log \|L(t)\xi_i^0\|.$$

We first prove the equivalence of the definitions of Lyapunov exponents given by (1.2.17) and (1.2.18), and then consider under what condition the exponents exist.

Lemma 1.2.1 *The definitions of Lyapunov exponents given by* (1.2.17) *and* (1.2.18) *are equivalent.*

Proof Note that

$$\langle L\xi_i, L\xi_j\rangle \;=\; \langle L^*L\xi_i, \xi_j\rangle \;=\; \alpha_i^2 \langle \xi_i, \xi_j\rangle \;=\; \delta_{i,j}\alpha_i^2,$$

to find the result.□

Definition 1.2.28 The linear system (1.2.14) (or (1.2.15)) is said to be *regular* if for some orthonormal basis $\{\xi_i\}$ of X

$$\sum_i \mu_i = \lim_{t\to\infty} \frac{1}{t} \log \left| \det \left[\left\langle L(t)\xi_j^0, L(t)\xi_k^0 \right\rangle \right]_{j,k=1,2,3\cdots} \right|,$$

for $t \in \mathbb{R}^+$ (or $t \in \mathbb{N}$).

In Definition 1.2.27 the existence of the orthonormal set of eigenvectors and existence of the limit is guaranteed for almost all U^0 (with respect to a suitable measure) by the *Multiplicative Ergodic Theorem* due to Osceledec [103] in $\mathbb{R}^p$ and for almost all U^0 (in the appropriate measure) in a separable Hilbert space with L compact by [113] provided the system (1.2.14) (or (1.2.15)) is regular.

Notes:

- We follow standard convention and order the Lyapunov exponents so that

$$\mu_1 \geq \mu_2 \geq \mu_3 \geq \cdots .$$

- In general it is only true that

$$\sum_i \mu_i \geq \limsup_{t\to\infty} \frac{1}{t} \log \left| \det \left[\left\langle L(t)\xi_j^0, L(t)\xi_k^0 \right\rangle \right]_{j,k=1,2,\cdots} \right| ; \tag{1.2.19}$$

  for $t > 0$ or $t \in \mathbb{N}$, see for example [116] or[113].

- For $X = \mathbb{R}^p$ one may show for the continuous system (1.2.14) that

$$\limsup_{t\to\infty} \frac{1}{t} \log \left| \det \left[\left\langle L(t)\xi_j^0, L(t)\xi_k^0 \right\rangle \right]_{j,k=1,2\cdots} \right| = \limsup_{t\to\infty} \frac{1}{t} \int_0^t \mathrm{Tr}\,(A(s))\ ds$$

  and so a continuous system in $\mathbb{R}^p$ is regular provided

$$\sum_i \mu_i = \limsup_{t\to\infty} \frac{1}{t} \int_0^t \mathrm{Tr}\,(A(s))\ ds.$$

  The proof of this may be found in [116].

Example Our first example is of a non–regular system which originates with Lyapunov [93] and is also treated in [116, p. 462]. Numerically it is considered in [36].

Consider the system

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cos(\log(t+1)) & \sin(\log(t+1)) \\ \sin(\log(t+1)) & \cos(\log(t+1)) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \forall t > 0. \tag{1.2.20}$$

Then,

$$\limsup_{t\to\infty} \frac{1}{t} \int_0^t Tr(A(s))ds = \sqrt{2},$$

and

$$\liminf_{t\to\infty} \frac{1}{t} \int_0^t Tr(A(s))ds = -\sqrt{2},$$

whereas (see [116])

$$\mu_1 + \mu_2 = 2.$$

Therefore the system is not regular.

We now consider the Lyapunov exponents for the linear evolution equations given by (1.2.5) and (1.2.9). In this case the Lyapunov exponents are a generalization of the linear stability analysis of Section 1.2.2. To illustrate this consider the following example.

Example Let $X = \mathbb{R}^p$ and consider the continuous system (1.2.1) and the linear stability equation given by (1.2.5). We recall the solution of (1.2.5) may be written as

$$\xi_i(t) = L(t, U^0)\xi_i^0 \quad i = 1, \cdots, p,$$

where $L(t, U^0) = e^{tDf[U(t)]}$ and $\xi_i^0 \in \mathbb{R}^p$.

The Lyapunov exponents $\{\mu_i\}_{i=1}^p$ are then given by

$$\begin{aligned} \mu_i &= \lim_{t\to\infty} \frac{1}{t} \log \|L(t)\xi_i^0\| \\ &= \lim_{t\to\infty} \frac{1}{t} \log \|e^{tDf[U(t)]}\xi_i^0\|. \end{aligned}$$

In the special case when $U(t)$ is a stationary solution $\widetilde{U}$ then the matrix $Df[U(t)] = Df[\widetilde{U}]$ is independent of t and the ith exponent is given by

$$\mu_i = \lim_{t\to\infty} \frac{1}{t} \log \|e^{tDf[\widetilde{U}]}\xi_i^0\|.$$

Letting $\{\lambda_i\}$ be the eigenvalues of $Df[\widetilde{U}]$ we find

$$\mu_i = \lim_{t\to\infty} \frac{1}{t} \log |e^{\lambda_i t}| = \lim_{t\to\infty} \frac{1}{t} \log |e^{Re[\lambda_i t]} e^{iIm[\lambda_i t]}|$$

$$= \lim_{t\to\infty} \frac{1}{t} \log |e^{Re[\lambda_i t]}| = \lim_{t\to\infty} \frac{1}{t} \mathrm{Re}[\lambda_i] t = \mathrm{Re}[\lambda_i].$$

Thus the Lyapunov exponents for a fixed point are exactly the real parts of the eigenvalues of $Df[\widetilde{U}]$ and we have the same linear stability analysis as in section 1.2.2. The stability of the fixed point depending on the sign of the eigenvalues of $Df[\widetilde{U}]$.

From the previous example we see that the ith Lyapunov exponent measures the average exponential rate of convergence or divergence of initial conditions separated an infinitesimal distance in the direction ξ_i^0. A zero Lyapunov exponent corresponds to pertubations along the solution; a positive exponent to exponential growth and a negative exponent to exponential decay of pertubations to the solution.

For a regular system the sum of m Lyapunov exponents gives the exponential rate of expansion or contraction of a $m-$volume element in X. For a regular system we have

$$\mu_1 + \mu_2 + \cdots + \mu_m = \limsup_{t\to\infty} \frac{1}{t} \log \left| \det \left[\left\langle L(t, U^0)\xi_j^0, L(t, U^0)\xi_k^o \right\rangle_{j,k=1,\cdots,m} \right] \right|. \quad (1.2.21)$$

which we can also write as

$$\mu_1 + \mu_2 + \cdots + \mu_m = \limsup_{t\to\infty} \frac{1}{t} \log \left| L(t, U^0)\xi_1^0 \wedge \cdots \wedge L(t, U^0)\xi_m^0 \right|, \quad (1.2.22)$$

where $\wedge$ denotes the standard exterior product (see section 3.5.3). The right hand side of (1.2.22) is exactly the volume of an m-volume element. Thus we find

μ_1	gives rate of	linear	expansion
$\mu_1 + \mu_2$	gives rate of	area	expansion
$\mu_1 + \mu_2 + \mu_3$	gives rate of	volume	expansion
$\mu_1 + \cdots + \mu_4$	gives rate of	4-volume	expansion
$\vdots$	$\vdots$	$\vdots$	$\vdots$

For $X = \mathbb{R}^p$ and if we assume that the system is ergodic (i.e. roughly speaking almost every trajectory visits all of the attractor) the dynamics on an attractor may be characterized by the signs of the associated Lyapunov exponents. This led to the introduction of the following notation due to Crutchfield [31])

$$(sign(\mu_1), sign(\mu_2), \cdots, sign(\mu_m))$$

to describe the asymptotic dynamics of stable attractors. Further details on Lyapunov exponents and ergodicity may be found in [44, 114].

For example in 3 dimensions we have the following symbolic representation of stable attractors:

$(-,$	$-,$	$-)$	stable fixed point
$(0,$	$-,$	$-)$	stable limit cycle
$(0,$	$0,$	$-)$	stable torus
$(+,$	$0,$	$-)$	stable chaotic attractor.

Recall the zero exponent corresponds to a perturbation tangent to or along the trajectory and so every stable time dependent configuration must have at least one zero exponent. For the chaotic attractor the + corresponds to exponential stretching in one direction giving the sensitivity to initial conditions, the 0 corresponds to perturbations tangent to the trajectory, and the − is the contraction that keeps the solutions bounded. For a more complete treatment we refer the reader to [44, 112, 114].

In order to avoid this ergodicity condition the notion of global Lyapunov exponents was introduced by Constantin and Foias [27].

Definition 1.2.29 Consider the continuous system (1.2.1) or the discrete system (1.2.2) and assume that there exists a global attractor $\mathcal{A}$. Then the *global Lyapunov exponents* are defined iteratively from

$$\mu_1 = \limsup_{t\to\infty} \frac{1}{t} \log \left\{ \sup_{U^0\in\mathcal{A}} \sup_{\substack{\xi_i^0\in X \\ \|\xi_i^0\|\le 1}} \|L(t,U^0)\xi_i^0\| \right\}$$

and

$$\mu_1 + \cdots + \mu_m = \limsup_{t\to\infty} \frac{1}{t} \log \left\{ \sup_{U^0\in\mathcal{A}} \sup_{\substack{\xi_i^0\in X \\ \|\xi_i\|\le 1}} \|L(t,U^0)\xi_1 \wedge \cdots \wedge L(t,U^0)\xi_m\| \right\}.$$

where $t \in \mathbb{R}^+$ for the continuous case, and $t \in \mathbb{N}$ for the discrete case.

Example We return to the Lorenz equations for this example which we recall below

$$\begin{aligned} x_t &= \sigma(y-x) \\ y_t &= rx - y - xz \\ z_t &= xy - \beta z. \end{aligned}$$

The linear evolution equation is then given for $\xi^0 \in \mathbb{R}^3$ by

$$\frac{d}{dt}\xi = \begin{pmatrix} -\sigma & \sigma & 0 \\ r-z & -1 & -x \\ y & x & -\beta \end{pmatrix} \xi(t).$$

Letting $A(t)$ denote the Jacobean matrix above we find that

$$\limsup_{t\to\infty} \frac{1}{t} \int_0^t \mathrm{Tr}(A(s))ds = -\sigma - 1 - \beta$$

and

$$\liminf_{t\to\infty} \frac{1}{t} \int_0^t \mathrm{Tr}(A(s))ds = -\sigma - 1 - \beta.$$

Therefore the Lorenz equations are regular provided

$$\sum_{i=1}^{3} \mu_i = -\sigma - 1 - \beta.$$

We note that since the sum of exponents for a regular system determines the rate of exponential volume expansion or contraction, for

$$-\sigma - 1 - \beta < 0$$

we expect exponential contraction of volume in the Lorenz equations.

Lyapunov exponents are also believed to be closely related to other global properties of attractors and in particular the dimension of the attractor.

Definition 1.2.30 (Hausdorff Dimension) Let $W \subset X$ be a compact set. We define the *Hausdorff Dimension* of W, $D_H = D_H(W)$ by

$$D_H(W) = \inf\left\{d > 0 | \mu_H^d(W) = 0\right\},$$

where

$$\mu_H^d(W) = \lim_{r\theta 0, r>0} \mu_{H,r}^d(W)$$

and

$$\mu_{H,r}^d(W) = \inf\left\{\sum_{i=1}^{k} r_i^d \,\middle|\, W \subset \bigcup_{i=1}^{k} B_i, B_i \text{ open balls in } H \text{ of radii } r_i \leq r\right\}.$$

Definition 1.2.31 (Lyapunov Dimension) Let $\{\mu_i\}$ be the global Lyapunov exponents for a global attractor $\mathcal{A}$. Then the Kaplan–Yorke or *Lyapunov dimension* D_L is defined by

$$D_L := m + \frac{\mu_1 + \cdots + \mu_m}{|\mu_{m+1}|} \tag{1.2.23}$$

where m is the smallest integer such that

$$\mu_1 + \cdots + \mu_m > 0$$

and

$$\mu_1 + \cdots + \mu_m + \mu_{m+1} < 0.$$

It is conjectured by Kaplan and Yorke that the Lyapunov dimension yields an upper bound on the Hausdorff dimension, i.e.

$$D_H < D_L$$

and for certain cases this has been proved to be true. Constantin and Foias [27] have proved the Hausdorff dimension is bounded above by the Lyapunov dimension for the 2D Navier–Stokes equation and this method was extended to certain classes of problems in [30] and [56].

The following section introduces two methods for numerically estimating Lyapunov exponents in $\mathbb{R}^p$.

1.2.4 $\cdots$ and Numerical Methods

We introduce in this section two basic methods for the numerical estimation of Lyapunov exponents of the continuous system (1.2.2) in $\mathbb{R}^p$. Given some initial conditions $\{\xi_i^0\}_1^p$, in order to find the Lyapunov exponents we essentially wish to integrate numerically the stability equation (1.2.5) over a long time interval $[0, \tau]$ and keep the norms $\|\xi_i(\tau)\|$, $i = 1, \cdots, p$. However this direct strategy will in general fail. In the presence of exponential growth (i.e. a positive exponent) in the system all the vectors $\{\xi_i\}$ will align themselves in the direction associated with the exponential growth and their norm will grow exponentially. Thus only one exponent could be found and numerical overflow would rapidly occur. This is discussed in [11, 12, 120, 57, 134] among others.

The most popular method for computing the Lyapunov exponents $\{\mu_i\}_{i=1}^m$, where $1 \leq m \leq p$ was suggested by Benettin et al [11, 12] and Shimada and Nagashima [120]. It is known as the *Standard Method* and essentially it combines the numerical integration of the stability equation with Gram–Schmidt re–orthonormalization applied periodically. Suppose we wished to find the approximate exponents $\{\mu_i(t)\}$ at time $t = \tau$, then a renormalization interval T is picked so that $\tau = rT$ with $r \in \mathbb{N}$. The stability equation (1.2.5) is integrated over T a total of r times. The norms from the Gram–Schmidt process are kept at each stage and the integration restarted with the orthonormalized vectors. The ith approximate Lyapunov exponent is then given by

$$\mu_i(\tau) = \mu_i(rT) \quad = \quad \frac{1}{rT} \log \|\xi_i(rT)\|. \tag{1.2.24}$$

The algorithm is presented below.

The Standard Method

1. Let $n = 0$
2. Pick orthonormal set $\{\xi_i^0\}_{i=1}^m$ $1 \leq m \leq p$
3. Pick re–orthonormalization interval T and $r > 0$
4. while $n < r$ do :

   ▷ Solve (1.2.5) on $[nT, (n+1)T]$, initial condition $\{\xi_i^n\}_{i=1}^m$

   ▷ Apply Gram–Schmidt to $\{\xi_i^n\}_{i=1}^m$ to get $\{\xi_i^{n+1}\}_{i=1}^m$

   ▷ $n = n + 1$
5. Find $\mu_i(rT)$ by (1.2.24) for $i = 1, \cdots, m$ or equivalently (see [11, 12])

$$\mu_i(rT) = \frac{1}{t}\sum_{k=1}^{r} \log \|\xi_i^k\|.$$

In implementation the choice of initial vectors $\{\xi_i^0\}$ is normally taken to be the standard basis for $\mathbb{R}^m$ and there are no reports of this approach failing in the literature :- the Gram–Schmidt process ensures that the directions and rates of growth are measured correctly. However there is general agreement that the choice of the re–orthonormalization interval T is a matter of trial and error and that a bad choice of T may cause the algorithm to fail. For further discussion on the standard method we refer the reader to [11, 12, 36, 44, 57, 120] and [135].

Goldhirsch et al [57] propose an alternative method, the *Differential Method* in which the re–orthonormalization interval T is made infinitesimal so that the Gram–Schmidt process is being constantly applied.

The Differential Method

▷ Let $\{\xi_i^0\}_{i=1}^m$, $1 \leq m \leq p$ be the orthonormal set of initial vectors.

▷ Solve over the interval $[0,\tau]$ the following system of equations,

$$\frac{d\xi_i}{dt} = Df[U]\xi_i - \frac{<\xi_i, Df[U]\xi_i>\xi_i}{<\xi_i,\xi_i>} - \sum_{j=1}^{i-1} \frac{(<\xi_i, Df[U]\xi_j> + <\xi_j, Df[U]\xi_i>)}{<\xi_j,\xi_j>}\xi_j \tag{1.2.25}$$

for all $i = 1, \cdots, m$.

▷ Find the ith approximate Lyapunov exponent $\mu_i(\tau)$ by evaluating:

$$\mu_i(\tau) = \frac{1}{\tau}\int_0^\tau <\xi_i(s), Df[U(s)]\xi_i(s)> ds \tag{1.2.26}$$

The derivation of this method is presented [57]. We simply note that the second term on the right hand side of (1.2.25) corresponds to the re–normalization and the summation term to the re–orthogonalization and note the following theorem.

Theorem 1.2.2 *Suppose we are given an orthonormal set of initial vectors* $\{\xi_i^0\}_{i=1}^m$ *for* $1 \le m \le p$ *so that*

$$\left\langle \xi_i^0, \xi_j^0 \right\rangle = \delta_{ij}.$$

Then the system of differential equations (1.2.25) *preserves orthnonormailty, that is*

$$\langle \xi_i(t), \xi_j(t) \rangle = \delta_{ij} \qquad \forall t > 0.$$

Proof The proof follows immediately from the derivation of the equations in Goldhirsch et al[57] or, alternatively, may be proved by induction. □

To implement the differential method as a numerical scheme the system of differential equations (1.2.25) have to be integrated numerically over the interval $[0,\tau]$ and the Lyapunov exponents estimated from (1.2.26) by a quadrature rule. This implementation is discussed in greater detail in Chapter 2.

Example

We re–examine the non–regular system (1.2.20) numerically, the results may be seen in Figure 1.1. This was produced by the differential method, implemented using the scheme of section 2.1.1.

We note that the computed Lyapunov exponents $\mu_{1,2}^c$ fail to converge but appear to oscillate between -1 and 1 and that the sum of exponents oscillates between -1 and 1 so that

$$-1 < \mu_1^c, \mu_2^c < 1 \text{ and } -\sqrt{2} < \mu_1^c + \mu_2^c < \sqrt{2}.$$

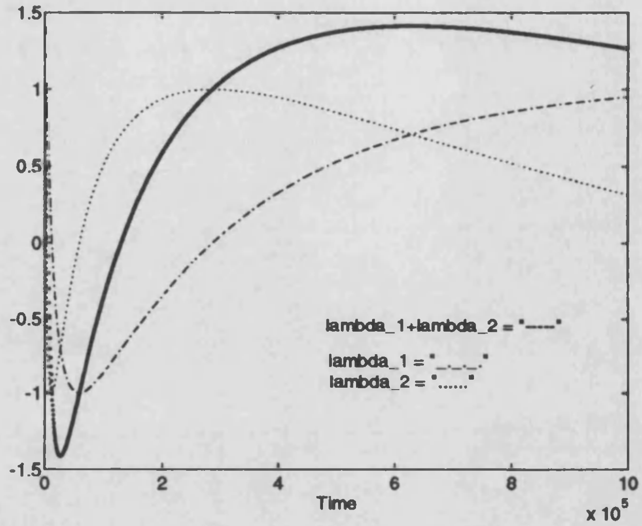

Figure 1.1: Lyapunov exponents and sum of exponents for a non–regular system.

These numerical results agree with those of [36] and are the same as those found using the standard method. They underline the fact that the numerical schemes are based on having a regular system.

1.3 Thesis Guide

Chapter 2:

A new numerical scheme is introduced for estimating Lyapunov exponents based on modifying a standard integration scheme to preserve orthonormality of initial vectors. A convergence proof is given. This is then extended to a general system of ordinary differential equations which conserves orthonormality. Numerical results are presented for the estimation of Lyapunov exponents for the Lorenz equations and our results compared with those of other authors.

Chapter 3:

The complex Ginzburg–Landau equation is introduced along with the mathematical framework. Standard results are reviewed such as the existence of a semi–group and the existence of a global attractor $\mathcal{A}$. We summarise results on the dimensionality of the global attractor and re–establish the existence of an inertial manifold using an existence proof based on attractive invariant manifolds in a Banach space.

We introduce the finite difference approximations to the Ginzburg–Landau equation that we consider in the thesis and lay down the discrete mathematical framework. We define norms on the vector space $\mathbb{C}^J$ and prove discrete versions of various continuous Sobolev space norm inequalities.

Chapter 4:

First we consider the complex Ginzburg–Landau equation discretized by finite differences in space. We prove that this semi–discrete system forms a continuous semi–group in the discrete L^2 space and that there are absorbing balls in both discrete L^2 and H^1 spaces of radius independent of the spatial step size. In section 4.2 we prove that solutions to the semi–discrete equations lie in a discrete Gevrey regularity class and hence prove the existence of a discrete Gevrey absorbing balls. Using the Gevrey regularity we are able to prove the main result for the semi–discrete equation: upper–semicontinuity of the semi–discrete global attractor $\mathcal{A}_{\Delta x}$ to the true global attractor $\mathcal{A}$.

A fully implicit approximation to the Ginzburg–Landau equation is analyzed in section 4.4 and we prove similar results to those for the semi–discrete approximation. In section 4.5 we analyse a mixed scheme where the non–linearity is treated in a explicit/implicit fashion. The chapter concludes with some numerical results and comments.

Chapter 5:

We prove in this chapter that the semi–discrete system and the fully implicit approximation admit an inertial manifold and that the cone condition is satisfied for both approximations. The Lyapunov dimension is estimated analytically for the semi–discrete system.

Chapter 6:

The complex Ginzburg–Landau equation admits many forms of exact solution. We look for discrete rotating wave solutions of our approximations perform a linear stability analysis and relate the results to the continuous problem. Neutral stability curves are determined analytically and investigated numerically.

Chapter 7:

In this chapter we briefly examine heteroclinic connections for the Ginzburg–Landau equation with periodic and Dirichlet boundary conditions using a spectral approximation.

Chapter 2

Numerical Schemes for Lyapunov Exponents

Although a number of people have proposed schemes for finding Lyapunov exponents [11, 12, 57, 120, 134, 135] and many people have been computing Lyapunov exponents (see for example [84, 25] or [129]) it is only recently that there has been any rigorous analysis of the numerical schemes employed. Dieci et al [36] present the most comprehensive analysis and that is for the particular case when all the exponents are calculated simultaneously.

Consider the evolution equation in $\mathbb{R}^p$ given by

$$U_t = F(U) \qquad\qquad U^0 = U(0) \text{ given,} \tag{2.0.1}$$

and the corresponding linear evolution equation

$$\xi_t = DF[U(t)]\xi \qquad\qquad \xi^0 = \xi(0) \text{ given.} \tag{2.0.2}$$

We are interested in estimating the set of Lyapunov exponents $\{\mu_1, \mu_2, \cdots, \mu_p\}$ or some subset

$$\{\mu_1, \mu_2, \cdots, \mu_k\}, \quad k \leq p$$

for the system (2.0.1) from the linear system (2.0.2).

For the purposes of this chapter we assume that the non–linear evolution system (2.0.1) is solved exactly. This is a common assumption in the literature, and in particular is also assumed by Dieci et al [36].

For the *Standard Method* (introduced in section 1.2.4) Dieci et al prove that

$$|\widetilde{\mu}_i - \hat{\mu}_i| \leq K_i\delta_i$$

where $\widetilde{\mu}_i$ is the Lyapunov exponent from the exact linear equation (2.0.2), $\hat{\mu}_i$ is the numerical approximation from (2.0.2), K_i is a constant and δ_i is a bound on the local error i.e. for $t \in [0, \tau]$, $t = n\Delta t$

$$||\xi_i(t) - \xi_i^n|| \leq \delta_i(\tau),$$

where ξ_i^n is the numerical approximation to $\xi_i(t)$.

For positive exponents they find that the standard method gives an accurate approximation to the exponents; however the computation of large negative exponents has been found to be difficult (see for example [115]). Dieci et al [36] note that their constant K_i becomes large for large negative exponents, which is consistent with the numerical results.

For the *Differential Method* introduced in section 1.2.4 it is essential that it is implemented correctly. If the orthogonality property of the differential equations (1.2.25) is not conserved then the scheme will fail. In Dieci et al the differential method is assumed to be implemented using one of the p^{th} order unitary methods (i.e. orthogonality preserving methods) in [35], and a composite trapezoidal rule for the integral (1.2.26). For this they prove that

$$|\widetilde{\mu}_i - \hat{\mu}_i| = O(\Delta t^2),$$

where $\widetilde{\mu}_i$ and $\hat{\mu}_i$ are as above.

A disadvantage of the analysis that Dieci et al perform is that they require all of the Lyapunov exponents be computed. For most applications one is only interested in computing the only the first few or the non-negative Lyapunov exponents. Furthermore for a large system, such as one arising from the discretization of a partial differential equation it is often computationally expensive to calculate all the exponents.

In section 2.1 we present a numerical scheme which ensures that the differential method is implemented in such a way that the norm and orthogonality of vectors is preserved and we prove convergence of this integration scheme. This scheme allows the set of exponents

$$\{\mu_1, \cdots, \mu_k\}, \quad k \le p,$$

to be calculated. In section 2.2 we generalize the numerical integration scheme to any set of ordinary differential equations which conserves the norm and orthogonality. Finally in section 2.4 numerical results are presented for the Lorenz equations and our results are compared with those of other authors. First, however, we introduce some notation and norms we shall use in this chapter and present the theory we invoke to prove convergence of the schemes.

Notation and Norms

- To simplify the equations for this chapter, and this chapter only, we shall use $J = J(t)$ to denote the Jacobian $DF[U(t)]$.
- We let $\Delta t > 0$ denote our time step.
- Let $\langle \bullet, \bullet \rangle$ denote an arbitrary inner product on $\mathbb{R}^p$ and let $\| \bullet \|$ be the induced norm so that $\| \bullet \|^2 = \langle \bullet, \bullet \rangle$.
- Let $v \in \mathbb{R}^{p \times N}$ where $v = (v^0, \cdots, v^{N-1})$ with $v^n \in \mathbb{R}^p$ and $\Delta t > 0$ be given. Then we define the following norms on $\mathbb{R}^{p \times N}$

$$\|v\|_1 := \max_n \|v^n\|, \quad \|v\|_2 := \|v^0\| + \Delta t \sum_{n=1}^{N-1} \|v^n\|.$$

Reconsider equation (2.0.1) and suppose the solution at time t is given by $U(t) = S(t)U^0$, where $S(t), t \ge 0$ is a semi–group of operators. Furthermore consider the mapping given by

$$U^{n+1} = S^{n+1}U^0, \text{ given } U^0, \tag{2.0.3}$$

where $\{S^n\}_{n \in \mathbb{N}}$ is a semi–group of operators.

Definition 2.0.1 The *truncation error* η for the map (2.0.3) as an approximation to (2.0.1) is defined as

$$\eta := S(\Delta t)U^0 - S^1 U^0.$$

The mapping (2.0.3) is said to be a *consistent* approximation to (2.0.1) of order Δt^r if there exists constant K such that $\|\eta\| \le K \Delta t^{r+1}$.

Definition 2.0.2 Let $h > 0$ be a real parameter, let $U_h \in \mathbb{R}^{p \times N}$ and let $\Phi : \mathbb{R}^{p\times N} \to \mathbb{R}^{p \times N}$ be dependent on h. Consider the *discretization*

$$\Phi(U_h) = 0. \tag{2.0.4}$$

Then (2.0.4) is said to be *k-stable* if there exists constants $K, h_0 > 0$ and $R \in (0, \infty)$ such that $\forall\ h < h_0$, and for all $V_h, W_h \in B(U_h, R) = \left\{w \in \mathbb{R}^{p\times N} : \|w - U_h\| < R\right\}$ we have that

$$\|V_h - W_h\| \leq K\|\Phi(V_h) - \Phi(W_h)\|.$$

Theorem 2.0.1 *If a numerical scheme is both consistent and k-stable then it is convergent.*

Proof See for example [117, 118].□

2.1 Numerical Implementation of the Differential Method

To start we briefly recall the differential method for estimating Lyapunov exponents. We shall assume for this section we are given initial vectors $\{e_i^0\}_{i=1}^p$ and constant C such that

$$\left\langle e_i^0, e_j^0 \right\rangle = C\delta_{i,j}\ \forall i = 1, \cdots, p. \tag{2.1.1}$$

Without loss of generality we take $C = 1$. The differential method for calculating the first k Lyapunov exponents consists of integrating

$$\frac{de_i}{dt} = Je_i - \frac{< e_i, Je_i > e_i}{< e_i, e_i >} - \sum_{j=1}^{i-1} \frac{(< e_i, Je_j > + < e_j, Je_i >)}{< e_j, e_j >} e_j\ \ \forall i = 1, \cdots, k,\ k \leq p \tag{2.1.2}$$

from 0 to t and then the ith time–dependent Lyapunov exponent is found by evaluating the integral

$$\mu_i(t) = \frac{1}{t}\int_0^t < e_i(s), J(s)e_i(s) > ds. \tag{2.1.3}$$

In section 1.2.4 we noted that the differential equations (2.1.2) conserve the norm and orthogonality of initial starting vectors, so that for initial vectors satisfying (2.1.1) we have that

$$\langle e_i(t), e_j(t)\rangle = \delta_{i,j}\ \forall t \geq 0.$$

To estimate the exponents computationally, we numerically integrate (2.1.2) for $t \in [0, T]$ and estimate the integral (2.1.3) by a quadrature rule.

If we were to try and solve the differential equations (2.1.2) using an arbitrary convergent numerical integration scheme, then we would probably fail to estimate the Lyapunov exponents accurately since a standard scheme does not conserve the orthonormality of initial vectors. In general all the vectors would align themselves in the direction associated with the largest exponent μ_1 and the norm of the vectors would not be conserved.

For example, suppose we naively approximate (2.1.2) for $i = 1, 2$ by explicit Euler's method to get

$$\frac{e_1^{n+1} - e_1^n}{\Delta t} = Je_1^n - \frac{\langle e_1^n, Je_1^n \rangle}{\|e_1^n\|^2} e_1^n$$

and

$$\frac{e_2^{n+1} - e_2^n}{\Delta t} = Je_2^n - \frac{\langle e_2^n, Je_2^n \rangle}{\|e_2^n\|^2} e_2^n - \frac{\langle e_2^n, Je_1^n \rangle + \langle e_1^n, Je_2^n \rangle}{\|e_1^n\|^2} e_1^n,$$

with $J = J(n\Delta t)$.

By considering e_1^{n+1} we see that the norm is not conserved

$$\begin{aligned} \|e_1^{n+1}\|^2 &= \langle e_1^{n+1}, e_1^n + \Delta t J e_1^n - \Delta t \langle e_1^n, Je_1^n \rangle \|e_1^n\|^{-2} e_1^n \rangle \\ &= \|e_1^n\|^2 + \Delta t^2 \langle Je_1^n - \langle e_1^n, Je_1^n \rangle \|e_1^n\|^{-2} e_1^n, Je_1^n - \langle e_1^n, Je_1^n \rangle \|e_1^n\|^{-2} e_1^n \rangle . \end{aligned}$$

Indeed we have found that

$$\|e_1^{n+1}\| = \|e_1^n\| + O(\Delta t).$$

This error can have a dramatic effect and this is illustrated in Figure 2.1 where an explicit Euler method has been used to solve (2.1.2) in the case when (2.0.1) is the Lorenz equations (solved using the standard fourth order Runge–Kutta method). We observe that the norm of the first vector e_1 appears to grow exponentially. We note that it can also be shown that orthogonality is not conserved by explicit Euler's method.

Our approach for finding the Lyapunov exponents is simple and reliable: we use a standard scheme combined with a projection so that a Gram–Schmidt process is applied at each step. It is illustrated below by choosing the explicit Euler method as our standard scheme. The integral (2.1.3) is estimated by a standard quadrature rule such as the trapezoidal rule.

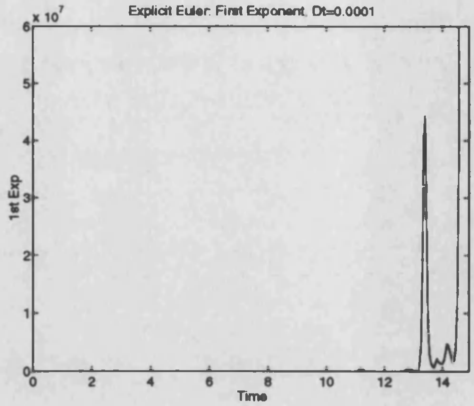

Figure 2.1: First exponent for Lorenz equations by explicit Euler's method ($r = 45.92$)

For implementation we have to choose some initial set of starting vectors $e_i(0), i = 1, \ldots, p$. If calculating all exponents then the standard basis for $\mathbb{R}^p$ would be adequate as a choice for $e_i(0), i = 1, \ldots, p$. However if only a subset of the Lyapunov exponents is required, for example just the largest exponent, then it is advisable to pick initial vectors $e_i(0)$ $i = 1, \ldots, q$, $(q \leq p)$ which are non-zero in every component. This guarantees that all directions of growth are considered.

In section (2.1.1) we introduce the scheme and prove that it conserves orthonormality of initial vectors. In section (2.1.2) we prove convergence of the scheme to the differential equations (2.1.2) using Theorem 2.0.1 based on *consistency* and *k-stability* [117, 118]. Finally we examine convergence of the Lyapunov exponent found numerically to the Lyapunov exponent found from the linear system (1.2.14).

2.1.1 The Scheme

The idea behind this scheme is to solve (2.0.2) using a standard method and to project to conserve orthonormality.

Let $S(t)$ be the semi-group associated with (2.0.2) so that,

$$\xi_i(t) = S(t)\xi_i^0;$$

let $\widetilde{S}(t)$ be the semi-group associated with (2.1.2) and let S^1 be the semi-group of an arbitrary convergent numerical method of order r. Thus there exists a constant $C \in \mathbb{R}$, independent of n, such that

$$\xi_i^{n+1} = S^1 \xi_i^n$$

and with $t = (n+1)\Delta t$

$$\|S(t)\xi_i^0 - S^1 \xi^n\| \leq C\Delta t^r.$$

Let e_i^{**} be given by

$$e_i^{**} := S_i^1 e_i^n - \sum_{j=1}^{i-1} \frac{\langle S^1 e_i^n, e_j^{n+1}\rangle}{\|e_j^{n+1}\|^2} e_j^{n+1}, \tag{2.1.4}$$

and define e_i^{n+1} by

$$e_i^{n+1} := e_i^{**} \frac{\|e_i^n\|}{\|e_i^{**}\|}. \tag{2.1.5}$$

Then the mapping $\widetilde{S}^1$ defined by

$$e_i^{n+1} = \widetilde{S}^1 e_i^n$$

preserves the orthonormality of the e_i's. We prove this statement in the following theorem.

Theorem 2.1.1 *Suppose we are given* $\{e_i^0\}_{i=1}^p$ *satisfying* (2.1.1). *Then the scheme (2.1.4,2.1.5) conserves orthonormality, that is*

$$\langle e_i^0, e_k^0\rangle = \delta_{i,k} \implies \left\langle e_i^n, e_j^n \right\rangle = \delta_{i,j} \ \forall n \geq 0.$$

Proof This is proved by induction on n, and we present here only the inductive step.

• First consider the case $i = k$. Suppose e_i^n is known and satisfies $\|e_i^n\| = \|e_i^0\|$ for all $i = 1, \cdots, p$. Then

$$\|e_i^{n+1}\|^2 = \langle e_i^{n+1}, e_i^{n+1}\rangle = \langle e_i^{**}, e_i^{**}\rangle \frac{\|e_i^n\|^2}{\|e_i^{**}\|^2} =< e_i^n, e_i^n >= \|e_i^0\|^2.$$

• For the case $i \neq k$ assume without loss of generality that $k < i$ and consider the inner product

$$< e_i^{n+1}, e_k^{n+1} > \ = \ < e_i^{**}, e_k^{n+1} > \frac{\|e_i^n\|}{\|e_i^{**}\|}.$$

We now induct on i and k.

Induction I : we prove that $\langle e_i^{n+1}, e_1^{n+1}\rangle = 0 \; \forall i \geq 2$.

The case $i = 2$ is straightforward to show. Assuming **I** true for $i-1$ we prove **I** true for i by noting that

$$\langle e_i^{n+1}, e_1^{n+1}\rangle = \left\{ \langle S^1 e_i^n, e_1^{n+1}\rangle - \sum_{j=1}^{i-1} \frac{\langle S^1 e_i^n, e_j^{n+1}\rangle}{\|e_j^{n+1}\|^2} \left\langle e_j^{n+1}, e_1^{n+1} \right\rangle \right\} \frac{\|e_i^n\|}{\|e_i^{**}\|}.$$

By the inductive hypothesis of **I** we get that

$$\begin{aligned} \langle e_i^{n+1}, e_1^{n+1}\rangle &= \langle S^1 e_i^n, e_1^{n+1}\rangle - \langle S^1 e_i^n, e_1^{n+1}\rangle \\ &= 0, \end{aligned}$$

and the induction on i is complete.

Induction K : we prove that $\langle e_i^{n+1}, e_k^{n+1}\rangle = 0 \;\; \forall i \geq 0, \; \forall k < i$. First note this is true for $k = 1$ by **Induction I.** Assuming **K** true for $k < i-2$. Then for $k = i-1$ we have

$$\langle e_i^{n+1}, e_{i-1}^{n+1}\rangle = \left\{ \langle S^1 e_i^n, e_{i-1}^{n+1}\rangle - \sum_{j=1}^{i-1} \frac{\langle S^1 e_i^n, e_j^{n+1}\rangle}{\|e_j^{n+1}\|^2} \left\langle e_j^{n+1}, e_{i-1}^{n+1} \right\rangle \right\} \frac{\|e_i^n\|}{\|e_i^{**}\|}$$

which by the inductive hypothesis **K** becomes

$$\begin{aligned} \langle e_i^{n+1}, e_1^{n+1}\rangle &= \langle S^1 e_i^n, e_1^{n+1}\rangle - \langle S^1 e_i^n, e_1^{n+1}\rangle \\ &= 0. \end{aligned}$$

Thus our proof of **K** is complete.

This concludes the induction on n and the theorem is proved. □

We shall assume that the Jacobian $J = J(n\Delta t)$ is evaluated exactly and that J is a bounded linear operator.

It would be possible to prove that the scheme (2.1.4, 2.1.5) converges to (2.1.2) as $\Delta t \to 0$ by proving that the projection (or Gram–Schmidt process) was of order Δt^r. We shall illustrate the convergence proof by taking as our standard method the explicit Euler method for which

$$S^1 e_i^n = e_i^n + \Delta t J e_i^n.$$

Once we have proved convergence we shall have a reliable method for calculating Lyapunov exponents.

2.1.2 Convergence of the Scheme

We aim to prove convergence of (2.1.4, 2.1.5) to (2.1.2) for the case when (2.0.2) is discretized by an explicit Euler method, other cases can be treated similarly. Thus we consider

$$S^1 e_i^n = e_i^n + \Delta t J e_i^n.$$

In which case 2.1.4 and 2.1.5 become

$$e_i^{**} = e_i^n + \Delta t J e_i^n - \sum_{j=1}^{i-1} \frac{< e_i^n + \Delta t J e_i^n, e_j^{n+1} >}{< e_j^{n+1}, e_j^{n+1} >} e_j^{n+1}; \tag{2.1.6}$$

and

$$e_i^{n+1} = e_i^{**} \frac{\|e_i^n\|}{\|e_i^{**}\|} \tag{2.1.7}$$

respectively. We now re–write (2.1.6,2.1.7) in a more familiar form: one that looks like a discretized ordinary differential equation. Consider

$$\frac{e_i^{n+1} - e_i^n}{\Delta t} = f_i(e_i^n) \tag{2.1.8}$$

where $i = 1, \cdots, p$, and $f_i : \mathbb{R}^{p \times i} \to \mathbb{R}^p$ is given for $w_i \in \mathbb{R}^p$, $\{w_j^{n+1}\}_{j=1}^{i-1}$, $w_j \in \mathbb{R}^p$ by

$$\begin{aligned} f_i(w_i) &:= f(w_i; w_{i-1}^{n+1}, w_{i-2}^{n+1}, \cdots, w_1^{n+1}) \\ &= \Big\{ (\|w_i^n\| - \|w_i^{**}\|)\, w_i^n + \Delta t \|w_i^n\| J w_i^n \\ &\quad - \|w_i^n\| \sum_{j=1}^{i-1} \frac{< w_i^n + \Delta t J w_i^n, w_j^{n+1} >}{< w_j^{n+1}, w_j^{n+1} >} w_j^{n+1} \Big\} \times \frac{1}{\Delta t \|w_i^{**}\|} . \end{aligned} \tag{2.1.9}$$

We extend our notation in order to define the discretization. Let $\alpha \in \mathbb{R}$ and let $\mathbf{e} \in \mathbb{R}^{p \times N}$, $\mathbf{e} = (e_i^0, e_i^1, \ldots, e_i^{N-1})$. Then we define the discretization of the ith equation $\Phi_i : \mathbb{R}^{p \times N} \to \mathbb{R}^{p \times N}$ by

$$\Phi_i(\mathbf{e}) = \begin{pmatrix} e_i^0 & - & \alpha \\ \frac{e_i^1 - e_i^0}{\Delta t} & - & f_i(e_i^1) \\ \vdots & & \\ \frac{e_i^{N-1} - e_i^{N-2}}{\Delta t} & - & f_i(e_i^{N-2}) \end{pmatrix} = \begin{pmatrix} \Phi_i(\mathbf{e})^0 \\ \Phi_i(\mathbf{e})^1 \\ \vdots \\ \Phi_i(\mathbf{e})^{N-1} \end{pmatrix}. \tag{2.1.10}$$

In order to prove the convergence we use the "consistency + k–stability" result of Theorem 2.0.1 detailed in either [117] or [118]. First we work towards the consistency result for which we require the following lemma.

Lemma 2.1.1 *Given initial vectors* $\{e_i^0\}_{i=1}^p$ *satisfying* (2.1.1), *suppose that* $\{e_i^n\}_{i=1}^p$ *is found from* (2.1.8). *Then, for all* $i = 1, \cdots, p$, *there exists* $C_i = C_i(Je_i^n, e_{i-1}^n, Je_{i-1}^n, \cdots, Je_1^n, e_1^n)$ *such that*

$$\|e_i^{**}\|^2 = \|e_i^n\|^2 + 2\Delta t \langle e_i^n, Je_i^n\rangle + C_i \Delta t^2. \tag{2.1.11}$$

Proof Take the inner product of e^{**} and e^{**} and expand to get

$$\begin{aligned}
\langle e_i^{**}, e_i^{**}\rangle &= \langle e_i^n, e_i^n\rangle + 2\Delta t \langle e_i^n, Je_i^n\rangle - 2\sum_{j=1}^{i-1} \frac{\langle e_i^n + \Delta t Je_i^n, e_j^{n+1}\rangle}{\langle e_j^{n+1}, e_j^{n+1}\rangle} \left\langle e_i^n, e_j^{n+1}\right\rangle \\
&\quad + \Delta t^2 \langle Je_i^n, Je_i^n\rangle - 2\Delta t \sum_{j=1}^{i-1} \frac{\langle e_i^n + \Delta t Je_i^n, e_j^{n+1}\rangle}{\langle e_j^{n+1}, e_j^{n+1}\rangle} \left\langle Je_i^n, e_j^{n+1}\right\rangle \\
&\quad + \sum_{j=1}^{i-1} \frac{\langle e_i^n + \Delta t Je_i^n, e_j^{n+1}\rangle^2}{\langle e_j^{n+1}, e_j^{n+1}\rangle^2} \left\langle e_j^{n+1}, e_j^{n+1}\right\rangle \\
&\quad + \sum_{r,s\ r\neq s}^{i-1} \frac{\langle e_i^n + \Delta t Je_i^n, e_r^{n+1}\rangle}{\langle e_r^{n+1}, e_r^{n+1}\rangle} \frac{\langle e_i^n + \Delta t Je_i^n, e_s^{n+1}\rangle}{\langle e_s^{n+1}, e_s^{n+1}\rangle} \langle e_r^{n+1}, e_s^{n+1}\rangle .
\end{aligned} \tag{2.1.12}$$

By inspection of (2.1.12), our proof will be complete if we can show that for all $j < i$

$$\left\langle e_i^n, e_j^{n+1}\right\rangle = \Delta t \left\langle e_i^n, Je_j^n\right\rangle + C_i \Delta t^2. \tag{2.1.13}$$

Proof of (2.1.13) : We prove equation (2.1.13) by induction on j and present here the inductive step. Assume that (2.1.13) holds for $1 \le j \le i-2$. Then for $j = i-1$ we have

$$\begin{aligned}
&\langle e_i^n, e_{i-1}^{n+1}\rangle \\
&= \left\{ \langle e_i^n, e_{i-1}^n\rangle + \Delta t \langle e_i^n, Je_{i-1}^n\rangle - \sum_{k=1}^{i-2} \frac{\langle e_{i-1}^n + \Delta t Je_{i-1}^n, e_k^{n+1}\rangle}{\langle e_k^{n+1}, e_k^{n+1}\rangle} \langle e_i^n, e_{i-1}^{n+1}\rangle \right\} \frac{\|e_{i-1}^n\|}{\|e_{i-1}^{**}\|}.
\end{aligned} \tag{2.1.14}$$

Let us look at each term on the right hand side of (2.1.14) separately. Since $\langle e_i^n, e_{i-1}^n\rangle = 0$ and by the inductive hypothesis the term in large curly brackets is of order Δt :

$$\begin{aligned}
&\left\{ \langle e_i^n, e_{i-1}^n\rangle + \Delta t \langle e_i^n, Je_{i-1}^n\rangle - \sum_{k=1}^{i-2} \frac{\langle e_{i-1}^n + \Delta t Je_{i-1}^n, e_k^{n+1}\rangle}{\langle e_k^{n+1}, e_k^{n+1}\rangle} \langle e_i^n, e_{i-1}^{n+1}\rangle \right\} \\
&\qquad = \Delta t \langle e_i^n, Je_{i-1}^n\rangle + C_{i-2}\Delta t.
\end{aligned}$$

For the last term of (2.1.14) we note that by the inductive hypothesis (2.1.11) holds for $1 \leq j \leq i-1$. If we apply Taylor's theorem we find

$$\begin{aligned}
\frac{\|e_{i-1}^n\|}{\|e_{i-1}^{**}\|} &= \left\{ \frac{\langle e_{i-1}^n, e_{i-1}^n \rangle}{\langle e_{i-1}^n, e_{i-1}^n \rangle + 2\Delta t \langle e_{i-1}^n, J e_{i-1}^n \rangle + C_{i-1}\Delta t^2} \right\}^{\frac{1}{2}} \\
&= \frac{\langle e_{i-1}^n, e_{i-1}^n \rangle^{1/2}}{\langle e_{i-1}^n, e_{i-1}^n \rangle^{1/2}} \left\{ 1 - \Delta t \frac{\langle e_{i-1}^n, J e_{i-1}^n \rangle}{\langle e_{i-1}^n, e_{i-1}^n \rangle} + C_{i-1}\Delta t^2 \right\} \\
&= 1 - \Delta t \frac{\langle e_{i-1}^n, J e_{i-1}^n \rangle}{\langle e_{i-1}^n, e_{i-1}^n \rangle} + C_{i-1}\Delta t^2.
\end{aligned}$$

Returning to (2.1.14) we have

$$\langle e_i^n, e_{i-1}^{n+1} \rangle = \Delta t \langle e_i^n, J e_i^n \rangle + C_{i-1}\Delta t^2, \tag{2.1.15}$$

and so (2.1.13) is proved.

Applying (2.1.15) to (2.1.12) we find (2.1.11).□

Corollary 2.1.1 *For all $i = 1, \cdots, p$ there exists $C_i = C_i(Je_i^n, e_i^n, \cdots, Je_1^n, e_1^n)$ such that we can expand $\|e_i^{**}\|$ and $\|e_i^{**}\|^{-1}$ as*

$$\|e_i^{**}\| = \|e_i^n\| \left\{ 1 + \Delta t \frac{\langle e_i^n, J e_i^n \rangle}{\|e_i^n\|^2} + C_i \Delta t^2 \right\} \tag{2.1.16}$$

and

$$\|e_i^{**}\|^{-1} = \|e_i^n\|^{-1} \left\{ 1 - \Delta t \frac{\langle e_i^n, J e_i^n \rangle}{\|e_i^n\|^2} + C_i \Delta t^2 \right\}. \tag{2.1.17}$$

Proof The proof in each case is an application of Taylor's theorem to (2.1.11).□

Theorem 2.1.2 (Consistency) *Let E_i^n denote the solution of the i^{th} differential equation of (2.1.2) at time $t = n\Delta t$. Then the local truncation error η_i^n satisfies:*

$$\eta_i^n := \frac{E_i^{n+1} - E_i^n}{\Delta t} - f_i(E_i^n) = O(\Delta t),$$

and hence the scheme is consistent of order Δt.

Proof We start by applying (2.1.13) to the summation term in (2.1.8)

$$\begin{aligned}
&\frac{E_i^{n+1} - E_i^n}{\Delta t} \\
&= f_i(E_i^n) + \eta_i^n \\
&= \left\{ (\|E_i^n\| - \|E_i^{**}\|) E_i^n + \Delta t \|E_i^n\| J E_i^n - \Delta t \|E_i^n\| \sum_{j=1}^{i-1} \frac{\langle E_i^n, J E_j^n \rangle + \langle J E_i^n, E_j^n \rangle}{\|E_j^{n+1}\|^2} E_j^{n+1} \right\} \\
&\quad \times \frac{1}{\Delta t \|E_i^{**}\|} + \eta_i^n + C_i \Delta t.
\end{aligned}$$

Now use the expansion (2.1.16) for $\|E_i^{**}\|$ inside the large brackets:

$$\begin{aligned}\frac{E_i^{n+1}-E_i^n}{\Delta t} &= \Bigg\{\|E_i^n\|JE_i - \frac{\langle E_i^n, JE_i^n\rangle}{\|E_i^n\|}E_i^n \\ &\quad -\|E_i^n\|\sum_{j=1}^{i-1}\frac{\left\langle E_i^n, JE_j^n\right\rangle + \left\langle JE_i^n, E_j^n\right\rangle}{\langle E_j^{n+1}, E_j^{n+1}\rangle}E_j^{n+1}\Bigg\}\|E_i^{**}\|^{-1} + \eta_i^n + C_i\Delta t.\end{aligned}$$

Finally use the expansion of $\|E_i^{**}\|^{-1}$ given by (2.1.17) to get

$$\begin{aligned}\eta_i^n &= \frac{E_i^{n+1}-E_i^n}{\Delta t} \\ &\quad -\left\{JE_i^n - \frac{\langle E_i^n, JE_i^n\rangle}{\langle E_i^n, E_i^n\rangle}E_i^n - \sum_{j=1}^{i-1}\frac{\left\langle E_i, JE_j^n\right\rangle + \left\langle E_j^n, JE_i^n\right\rangle}{\langle E_j^{n+1}, E_j^{n+1}\rangle}E_j^{n+1}\right\} + C_i\Delta t,\end{aligned}$$

and the result follows. □

Before proving that the scheme is k–stable, we show that for each i the function f_i defined in (2.1.9) is locally Lipschitz continuous.

Theorem 2.1.3 *For all Δt sufficiently small the function $f_i : \mathbb{R}^{p\times i} \to \mathbb{R}^p$, $i = 1,\cdots,p$ defined by (2.1.9) is locally Lipschitz continuous and so for w_i, v_i in a ball of radius ρ, $B(\rho)\subset \mathbb{R}^p$, there exists a constant $C = C(\rho)$ such that*

$$\|f_i(w_i) - f_i(v_i)\| \le C\|w_i - v_i\| \quad \forall i = 1,\cdots,p.$$

Proof By the same analysis as Theorem 2.1.2 we find, using Lemma 2.1.1, that for all $w_i \in \mathbb{R}^p$ there exists a constant C_i such that

$$f_i(w_i) = F_i(w_i) + C_i\Delta t,$$

where

$$F_i(w_i) := Jw_i - \frac{\langle w_i, Jw_i\rangle}{\langle w_i, w_i\rangle}w_i - \sum_{j=1}^{i-1}\frac{\left\langle w_i, Jw_j^n\right\rangle + \left\langle w_j^n, Jw_i\right\rangle}{\langle w_j^{n+1}, w_j^{n+1}\rangle}w_j^{n+1}.$$

We choose Δt sufficiently small so that

$$\Delta t \le \|w_i - v_i\|.$$

Consider $F_i(w_i) - F_i(v_i)$ and take norms to get

$$\|F_i(w_i^n) - F_i(v_i)\| \;\le\; \|Jw_i - Jv_i\| + \left\|\frac{\langle w_i, Jw_i\rangle}{\|w_i\|^2}w_i - \frac{\langle v_i, Jv_i\rangle}{\|v_i\|^2}v_i\right\|$$

$$+\sum_{j=1}^{i-1} \|w_j^{n+1}\|^{-2} \left\| \left\langle w_i, Jw_j^n \right\rangle w_j^{n+1} - \left\langle v_i, Jw_j^n \right\rangle w_j^{n+1} \right\|$$
$$+\sum_{j=1}^{i-1} \|w_j^{n+1}\|^{-2} \left\| \left\langle w_j^n, Jw_i \right\rangle w_j^{n+1} - \left\langle w_j^n, Jv_i \right\rangle w_j^{n+1} \right\|. \tag{2.1.18}$$

We consider (2.1.18) term by term, making use of J being a bounded linear operator. For the first term we simply note that

$$\|Jw_i - Jv_i\| \le \|J\|\|w_i - v_i\|. \tag{2.1.19}$$

The final two terms are dealt with by noting that

$$\left(\left\langle w_i, Jw_j^n \right\rangle - \left\langle v_i, Jv_j^n \right\rangle\right) w_j^{n+1} = \left\langle w_i - v_i, Jw_j^n \right\rangle w_j^{n+1},$$

and

$$\left(\left\langle w_j^n, Jw_i \right\rangle - \left\langle w_j^n, Jv_i \right\rangle\right) w_j^{n+1} = \left\langle w_j^n, J(w_i - v_i) \right\rangle;$$

so that

$$\left\| \left(\left\langle w_i, Jw_j^n \right\rangle - \left\langle v_i, Jv_j^n \right\rangle\right) w_j^{n+1} \right\| \le \|w_j^{n+1}\|\|w_j^n\|\|J\|\|w_i - v_i\|, \tag{2.1.20}$$

and

$$\left\| \left(\left\langle w_j^n, Jw_i \right\rangle - \left\langle w_j^n, Jv_i \right\rangle\right) w_j^{n+1} \right\| = \|w_j^{n+1}\|\|w_j^n\|\|J\|\|w_i - v_i\|. \tag{2.1.21}$$

The second term is dealt with as follows:

$$\begin{aligned} \left\| \frac{\langle w_i, Jw_i \rangle}{\|w_i\|^2} w_i - \frac{\langle v_i, Jv_i \rangle}{\|v_i\|^2} v_i \right\| \quad &\le \quad \frac{1}{\|w_i\|^2} \|\langle w_i, Jw_i \rangle w_i - \langle w_i, Jw_i \rangle v_i\| \\ &\quad + \left\| \frac{\langle w_i, Jw_i \rangle}{\|w_i\|^2} v_i - \frac{\langle v_i, Jv_i \rangle}{\|v_i\|^2} v_i \right\|. \end{aligned}$$

Assume, without loss of generality, that $\|w_i\| \ge \|v_i\|$ to get

$$\left\| \frac{\langle w_i, Jw_i \rangle}{\|w_i\|^2} w_i - \frac{\langle v_i - Jv_i \rangle}{\|v_i\|^2} v_i \right\| \tag{2.1.22}$$
$$\begin{aligned} &\le \quad \|J\|\|w_i - v_i\| + \|v_i\|^{-1} \|\langle w_i - v_i, Jw_i \rangle + \langle v_i, J(w_i - v_i) \rangle\| \\ &\le \quad \|J\|\|w_i - v_i\| + \|v_i\|^{-1}\|J\|\|w_i\|\|w_i - v_i\| + \|J\|\|w_i - v_i\| \end{aligned}$$
$$\le \quad 3\|J\|\|w_i - v_i\| \tag{2.1.23}$$

Combining equations (2.1.19–2.1.23) in (2.1.18) yields the desired result. □

Theorem 2.1.4 (stability) *Let E_i^n be the true solution of the i^{th} differential equation evaluated at $t = n\Delta t$ and define $E_i := (E_i^0, \cdots, E_i^{N-1})$. Let $B(E_i, r)$ be the open ball center E_i of radius r where $0 < r \leq \infty$. Suppose we are also given $\mathbf{e}$, $\mathbf{a} \in I\!R^{p\times N}$, $\mathbf{e} = (e_i^0, \cdots, e_i^{N-1})$ and $\mathbf{a} = (a_i^0, \cdots, a_i^{N-1})$ defined in $B(E_i, r)$ both arising from the discretization* (2.1.10).

Then there exists a constant $M = M(B(E_i, r))$, independent of Δt and $n = 1, \cdots, N-1$, such that

$$\|\mathbf{e} - \mathbf{a}\|_1 \leq e^{N\Delta t M} \|\Phi_i(\mathbf{e}) - \Phi_i(\mathbf{a})\|_2.$$

Proof First we bound $\|e_i^{n+1} - a_i^{n+1}\|$ in terms of $\|\Phi_i(\mathbf{e})^{n+1} - \Phi_i(\mathbf{a})^{n+1}\|$. Note that

$$\begin{aligned}
\Phi_i(\mathbf{e})^0 &= e_i^0 - \alpha, \qquad \Phi_i(\mathbf{e})^{n+1} = \frac{e_i^{n+1} - e_i^n}{\Delta t} - f_i(e_i^n) \\
\Phi_i(\mathbf{a})^0 &= a_i^0 - \alpha, \qquad \Phi_i(\mathbf{a})^{n+1} = \frac{a_i^{n+1} - a_i^n}{\Delta t} - f_i(a_i^n).
\end{aligned}$$

Subtract to get :

$$\Phi_i(\mathbf{e})^0 - \Phi_i(\mathbf{a})^0 = e_i^0 - a_i^0$$

and

$$e_i^{n+1} - a_i^{n+1} = (e_i^{n+1} - a_i^n) + \Delta t(f_i(e_i^n) - f_i(a_i^n)) + \Delta t(\Phi_i(\mathbf{e})^{n+1} - \Phi_i(\mathbf{a})^{n+1}) \quad (2.1.24)$$

Taking norms in equation (2.1.24) and using the local Lipschitz continuity of f we find

$$\|e_i^{n+1} - a_i^{n+1}\| \leq \|e_i^n - a_i^n\|(1 + \Delta t M) + \Delta t \|\Phi_i(\mathbf{e})^{n+1} - \Phi_i(\mathbf{a})^{n+1}\|.$$

Now using a standard induction argument and again taking norms we get

$$\begin{aligned}
\|\mathbf{e} - \mathbf{a}\|_1 &= \max_n \|e_i^n - a_i^n\| \\
&\leq e^{N\Delta t M}\left(\|\Phi_i(\mathbf{e})^0 - \Phi_i(\mathbf{a})^0\| + \sum_{n=1}^{N} \Delta t \|\Phi_i(\mathbf{e})^n - \Phi_i(\mathbf{a})^n\|\right) \\
&\leq e^{N\Delta t M} \|\Phi_i(\mathbf{e}) - \Phi_i(\mathbf{a})\|_2.
\end{aligned}$$

Hence the scheme is k-stable.□

Theorem 2.1.5 (Convergence) *The scheme* (2.1.8) *is convergent of $O(\Delta t)$ to the differential equations* (2.1.2) *for $i = 1, \cdots, p$.*

Proof By Theorem 2.1.2 the scheme (2.1.8) is consistent with order Δt to (2.1.2) and by Theorem 2.1.4 is k-stable. Hence by Theorem 2.0.1 the scheme is convergent. □

Figure 2.2 shows the Lyapunov exponents for the Lorenz equations found using the above scheme denoted ME (for modified Euler) with the integral (2.1.3) estimated by a quadrature rule - in this case the trapezoidal rule.

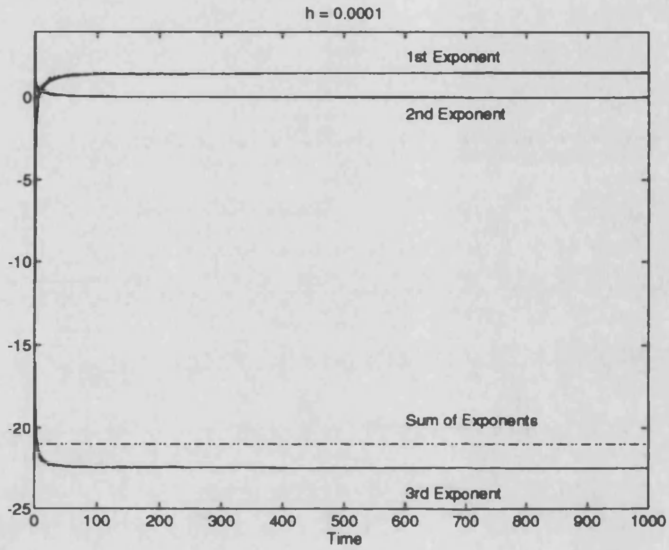

Figure 2.2: All 3 exponents for the Lorenz equations ($r = 45.92$) found by ME.

Figure 2.3 shows the corresponding computed solution to the Lorenz equations, what we observe is the typical "chaotic Lorenz attractor". Since the largest computed Lyapunov exponent is positive there is some indicatation of the presence of chaos.

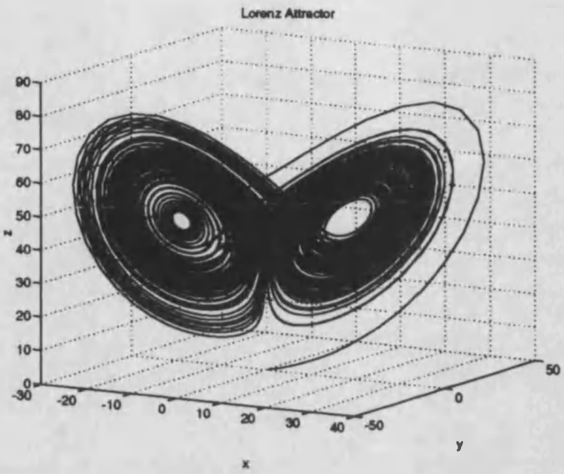

Figure 2.3: Numerical solution of Lorenz equations with $r = 45.92$ and $\Delta t = 0.0001$.

2.2 A Generalisation

We now consider any dynamical system where the flow preserves the orthonormality of the initial vectors. In particular we shall consider in this section the numerical approximation of the following problem in $\mathbb{R}^p$.

We shall assume throughout this section that we are given initial vectors $\{v_i^0\}_{i=1}^p$, where each $v_i \in \mathbb{R}^p$ and $C > 0$ such that

$$\left\langle v_i^0, v_j^0 \right\rangle = C\delta_{i,j} \ \forall i = 1, \cdots p. \tag{2.2.1}$$

Without loss of generality we take $C = 1$. Consider the following system of ordinary differential equations

$$\frac{dv_i}{dt} = g_i(v_i, v_{i-1}, \cdots, v_1) \quad i = 1, \cdots, p \tag{2.2.2}$$

subject to

$$\langle v_i, g_j(v_j, \cdots, v_1)\rangle + \langle g_i(v_i, \cdots, v_1), v_j\rangle \ = 0. \tag{2.2.3}$$

Let us assume the system (2.2.2) is a dynamical system and hence there exists a semi–group $S(t)$ $\forall$ $t \geq 0$ such that $v(t) = S(t)v^0$.

To simplify notation we omit the dependence of the functions g_i on v_j for $j < i$.

Theorem 2.2.1 *The system of ordinary differential equations* (2.2.2) *conserves the orthonormality of the initial vectors. That is, for all $t \geq 0$*

$$\left\langle v_i^0, v_j^0 \right\rangle = \delta_{i,j} \implies \langle v_i(t), v_j(t) \rangle = \delta_{i,j} \quad i,j = 1,\cdots,p. \tag{2.2.4}$$

Proof The proof is straightforward. Note that $\forall i,j$ we have that

$$\langle v_i(t), g_j(v_j(t)) \rangle + \langle g_i(v_i(t)), v_j(t) \rangle = \frac{1}{2}\frac{d}{dt} \langle v_i(t), v_j(t) \rangle = 0.$$

Thus for all $i, p = 1,\cdots,j$,

$$\langle v_i(t), v_j(t) \rangle = \left\langle v_i^0, v_j^0 \right\rangle \forall t \geq 0. \quad \square \tag{2.2.5}$$

We are interested in finding a numerical method to solve (2.2.2) which will preserve the conservation of orthogonormality.

As in Section 2.1 we propose schemes which are based on arbitrary convergent numerical methods with an added projection. We introduce the scheme in a general context and then prove convergence by *consistency + k-stability* in the case when the numerical scheme is the explicit Euler method.

The Scheme

Let S^1 be the semi–group associated with an arbitrary convergent numerical method of order r. Then we let

$$v_i^{**} = S^1 v_i^n - \sum_{j=1}^{i-1} \frac{\langle S^1 v_i^n, v_j^{n+1} \rangle}{\langle v_j^{n+1}, v_j^{n+1} \rangle} v_j^{n+1} \quad \forall\, i = 1,\cdots,p \tag{2.2.6}$$

and define our approximation to (2.2.2) to be given by

$$v_i^{n+1} = v_i^{**} \frac{\|v_i^n\|}{\|v_i^{**}\|} \quad \forall\, i = 1,\cdots,p. \tag{2.2.7}$$

The summation term in (2.2.6) corresponds to a re–orthogonalization and choice of weighting in (2.2.7) to the re–normalization.

Theorem 2.2.2 *Given $\{v_i^0\}_{i=1}^p$ the scheme (2.2.6-2.2.7) conserves othonormality of initial vectors. That is,*

$$\langle v_i^0, v_k^0 \rangle = \delta_{i,j} \implies \langle v_i^n, v_k^n \rangle = \delta_{i,j}, \quad i,k = 1,\cdots,p.$$

Proof The proof is by induction on n exactly like the proof of Theorem 2.1.1 and so is omitted here. $\square$

2.2.1 Convergence

We shall prove convergence in the special case when (2.2.2) is discretized by explicit Euler's method, i.e. when

$$S^1 v_i^n = v_i^n + \Delta t g_i(v_i^n) v_i^n,$$

where we once again suppress the dependence of g on $v_{i-1}, \cdots, v_1$ for the ease of notation.

In this case the scheme (2.2.6-2.2.7) is given by

$$v_i^{**} = v_i^n + \Delta t g_i(v_i^n) - \sum_{j=1}^{i-1} \frac{\langle v_i^n + \Delta t g_i(v_i^n), v_j^{n+1}\rangle}{\langle v_j^{n+1}, v_j^{n+1}\rangle} v_j^{n+1} \quad \forall\, i = 1, \cdots, p \tag{2.2.8}$$

and

$$v_i^{n+1} = v_i^{**} \frac{\|v_i^n\|}{\|v_i^{**}\|} \quad \forall\, i = 1, \cdots, p. \tag{2.2.9}$$

Again we may re-write the scheme to look more like a discretized ordinary differential equation:

$$\frac{v_i^{n+1} - v_i^n}{\Delta t} = f_i(v_i^n)$$

where $i = 1, \cdots, p$, and given $w_i \in \mathbb{R}^p$, $\{w_j^{n+1}\}_{j=1}^{i-1}$, $w_j^{n+1} \in \mathbb{R}^p$, the function $f_i : \mathbb{R}^{p \times i} \to \mathbb{R}^p$ is given by

$$\begin{aligned} f_i(w_i) &:= f_i(w_i, w_{i-1}^{n+1}, w_{i-2}^{n+1}, \cdots, w_1^{n+1}) \\ &= \Big\{ \left(\|w_i^n\| - \|w_i^{**}\|\right) w_i^n + \Delta t \|w_i^n\| g_i(w_i^n) \\ &\quad - \|w_i^n\| \sum_{j=1}^{i-1} \frac{< w_i^n + \Delta t g_i(w_i^n), w_j^{n+1} >}{< w_j^{n+1}, w_j^{n+1} >} w_j^{n+1} \Big\} \times \frac{1}{\Delta t \|w_i^{**}\|}. \end{aligned} \tag{2.2.10}$$

In order to prove convergence we invoke Theorem 2.0.1. We start with a lemma which is one step toward proving consistency.

Lemma 2.2.1 *Suppose that $\{v_i^n\}_{i=1}^p$ is found from (2.2.8-2.2.9). Then, for all $i = 1, \cdots, p$, there exists a constant $C_i = C_i(g_i(v_i^n), v_i^n, \cdots, g_1(v_1^n), v_1^n)$ such that*

$$\|v_i^{**}\|^2 = \|v_i^n\|^2 + C_i \Delta t^2.$$

Hence there exists $C_i = C_i(g_i(v_i^n), v_i^n, \cdots, g_1(v_1^n), v_1^n)$ such that

$$\frac{\|v_i^n\|}{\|v_i^{**}\|} = 1 + C_i \Delta t^2.$$

Proof Consider the expansion of $\langle v_i^{**}, v_i^{**}\rangle$ given by

$$\begin{aligned}
\langle v_i^{**}, v_i^{**}\rangle &= \langle v_i^n + \Delta t g_i(v_i^n), v_i^n + \Delta t g_i(v_i^n)\rangle \\
&\quad -2\left\langle v_i^n + \Delta t g_i(v_i^n), \sum_{j=1}^{i-1} \frac{\langle v_i^n + \Delta t g_i(v_i^n), v_j^{n+1}\rangle}{\langle v_j^{n+1}, v_j^{n+1}\rangle} v_j^{n+1}\right\rangle \\
&\quad +\left\langle \sum_{j=1}^{i-1} \frac{\langle v_i^n + \Delta t g_i(v_i^n), v_j^{n+1}\rangle}{\langle v_j^{n+1}, v_j^{n+1}\rangle} v_j^{n+1}, \sum_{j=1}^{i-1} \frac{\langle v_i^n + \Delta t g_i(v_i^n), v_j^{n+1}\rangle}{\langle v_j^{n+1}, v_j^{n+1}\rangle} v_j^{n+1}\right\rangle \\
&= \langle v_i^n, v_i^n\rangle + 2\Delta t \langle v_i^n, g_i(v_i^n)\rangle + \Delta t^2 \langle g_i(v_i^n), g_i(v_i^n)\rangle \\
&\quad -2\left\langle v_i^n + \Delta t g_i(v_i^n), \sum_{j=1}^{i-1} \frac{\langle v_i^n + \Delta t g_i(v_i^n), v_j^{n+1}\rangle}{\langle v_j^{n+1}, v_j^{n+1}\rangle} v_j^{n+1}\right\rangle \\
&\quad +\left\langle \sum_{j=1}^{i-1} \frac{\langle v_i^n + \Delta t g_i(v_i^n), v_j^{n+1}\rangle}{\langle v_j^{n+1}, v_j^{n+1}\rangle} v_j^{n+1}, \sum_{j=1}^{i-1} \frac{\langle v_i^n + \Delta t g_i(v_i^n), v_j^{n+1}\rangle}{\langle v_j^{n+1}, v_j^{n+1}\rangle} v_j^{n+1}\right\rangle .
\end{aligned} \tag{2.2.11}$$

By inspection of (2.2.11) our proof will be complete provided we can prove that for all $j < i$:

$$\left\langle v_i^n, v_j^{n+1}\right\rangle = \Delta t \left\langle v_i^n, g_j(v_j^n)\right\rangle + O(\Delta t^2). \tag{2.2.12}$$

Proof of (2.2.12) : We prove (2.2.12) by induction on j. The proof for $j = 1$ is straightforward. Assuming (2.2.12) is true for $1 \le j \le i-2$ we prove it is true for $j = i-1$.

$$\begin{aligned}
&\langle v_i^n, v_{i-1}^{n+1}\rangle \\
&= \left\{\langle v_i^n, v_{i-1}^n\rangle + \Delta t \langle v_i^n, g_{i-1}(v_{i-1}^n)\rangle - \sum_{k=1}^{i-2} \frac{\langle v_i^n + \Delta t g_i(v_i^n), v_k^{n+1}\rangle}{\langle v_k^{n+1}, v_k^{n+1}\rangle} \langle v_k^{n+1}, v_{i-1}^n\rangle\right\} \frac{\|v_{i-1}^n\|}{\|v_{i-1}^{**}\|}
\end{aligned} \tag{2.2.13}$$

By the inductive hypothesis we see that the term in large curly brackets in (2.2.13) can be re–written as

$$\begin{aligned}
&\left\{\langle v_i^n, v_{i-1}^n\rangle + \Delta t \langle v_i^n, g_{i-1}(v_{i-1}^n)\rangle - \sum_{k=1}^{i-2} \frac{\langle v_i^n + \Delta t g_i(v_i^n), v_k^{n+1}\rangle}{\langle v_k^{n+1}, v_k^{n+1}\rangle} \langle v_k^{n+1}, v_{i-1}^n\rangle\right\} \\
&\qquad = \Delta t \langle v_i^n, g_{i-1}(v_{i-1}^n)\rangle + C_{i-2}\Delta t^2.
\end{aligned} \tag{2.2.14}$$

We also see by the inductive hypothesis that

$$\|v_{i-1}^{**}\|^2 = \|v_{i-1}^n\|^2 + C_{i-1}\Delta t^2$$

and so (2.2.12) is established by induction on j.

Since we have now established (2.2.12) we have from (2.2.11) the first equality of the Theorem. An application of Taylor's theorem will give us the second inequality.□

We now aim to show that the re-orthogonalisation term (the summation term in (2.2.8)) is of order Δt^2.

Lemma 2.2.2

Given initial vectors $\{v_i^0\}_{i=1}^p$, suppose $\{v_i^n\}_{i=1}^p$ are found from (2.2.8 - 2.2.9). Then there exists $C_i = C_i(g_i(v_i^n), v_i^n, \cdots, g_1(v_1^n), v_1^n)$ such that

$$\left\langle v_i^n, g_j(v_j^n)\right\rangle = \frac{1}{2\Delta t}\left\{\left\langle v_i^{n+1}, v_j^n\right\rangle - \left\langle v_i^n, v_j^{n+1}\right\rangle\right\} + C_i\Delta t \;\forall\; i \geq j = 1, \cdots, p.$$

Proof

We can rearrange (2.2.8 - 2.2.9) and use Lemma 2.2.1 to get the following expressions for v_i^n and g_i^n:

$$v_i^n = \frac{1}{2}\left\{v_i^{n+1} + v_i^n - \Delta t g_i(v_i^n) + \sum_{k=1}^{i-1}\frac{\langle v_i^n + \Delta t g_i(v_i^n), v_k^{n+1}\rangle}{\langle v_k^{n+1}, v_k^{n+1}\rangle} v_k^{n+1}\right\} + C_i\Delta t^2 \qquad (2.2.15)$$

and

$$g_i(v_i^n) = \frac{1}{\Delta t}\left\{v_i^{n+1} - v_i^n + \sum_{k=1}^{i-1}\frac{\langle v_i^n + \Delta t g_i(v_i^n), v_k^{n+1}\rangle}{\langle v_k^{n+1}, v_k^{n+1}\rangle} v_k^{n+1}\right\} + C_i\Delta t. \qquad (2.2.16)$$

We now take the inner-product of v_i^n and $g_j(v_j^n)$ and use (2.2.15) and (2.2.16), multiplying by $2\Delta t$ to simplify the expression

$$\begin{aligned}
&2\Delta t\left\langle v_i^n, g_j(v_j^n)\right\rangle = \left\langle v_i^{n+1} + v_i^n, v_j^{n+1}\right\rangle - \left\langle v_i^{n+1} + v_i^n, v_j^n\right\rangle \\
&+\sum_{\ell=1}^{j-1}\frac{\left\langle v_j^n + \Delta t g_j(v_j^n), v_\ell^{n+1}\right\rangle}{\langle v_\ell^{n+1}, v_\ell^{n+1}\rangle}\langle v_\ell^{n+1}, v_i^{n+1} + v_i^n\rangle - \Delta t\left\langle g_i(v_i^n), v_j^{n+1}\right\rangle + \Delta t\left\langle g_i(v_i^n), v_j^n\right\rangle \\
&-\Delta t\sum_{\ell=1}^{j-1}\frac{\left\langle v_j^n + \Delta t g_j(v_j^n), v_\ell^{n+1}\right\rangle}{\langle v_\ell^{n+1}, v_\ell^{n+1}\rangle}\langle g_i(v_i^n), v_\ell^{n+1}\rangle + \sum_{k=1}^{i-1}\frac{\langle v_i^n + \Delta t g_i(v_i^n), v_k^{n+1}\rangle}{\langle v_k^{n+1}, v_k^{n+1}\rangle}\left\langle v_k^{n+1}, v_j^{n+1}\right\rangle \\
&-\sum_{k=1}^{i-1}\frac{\langle v_i^n + \Delta t g_i(v_i^n), v_k^{n+1}\rangle}{\langle v_k^{n+1}, v_k^{n+1}\rangle}\left\langle v_k^{n+1}, v_j^{n+1}\right\rangle \\
&+\left\langle\sum_{k=1}^{i-1}\frac{\langle v_i^n + \Delta t g_i(v_i^n), v_k^{n+1}\rangle}{\langle v_k^{n+1}, v_k^{n+1}\rangle}v_k^{n+1}, \sum_{\ell=1}^{j-1}\frac{\left\langle v_j^n + \Delta t g_j(v_j^n), v_\ell^{n+1}\right\rangle}{\langle v_\ell^{n+1}, v_\ell^{n+1}\rangle}v_\ell^{n+1}\right\rangle + C_i\Delta t.
\end{aligned}$$

Using Lemma 2.2.1 and equation (2.2.12) on each of the terms above, we see that

$$\left\langle v_i^n, g_j(v_j^n)\right\rangle = \frac{1}{2\Delta t}\left\{\left\langle v_i^{n+1}, v_j^n\right\rangle - \left\langle v_i^n, v_j^{n+1}\right\rangle\right\} + C_i\Delta t \qquad (2.2.17)$$

and the lemma is proved.□

Corollary 2.2.1 *There exists a constant $C_i = C_i(g_i(v_i^n), v_i^n, \cdots, g_1(v_1^n), v_1^n)$ such that for every $1 \le j < i \le p$*

$$\left\langle v_i^n, g_j(v_j^n)\right\rangle + \left\langle v_j^n, g_i(v_i^n)\right\rangle = C_i \Delta t^2. \tag{2.2.18}$$

Proof By Lemma 2.2.1 and equation (2.2.12) we have that

$$\left\langle v_i^n + \Delta t g_i(v_i^n), v_j^{n+1}\right\rangle = \Delta t \left\{\left\langle v_i^n, g_j(v_j^n)\right\rangle + \left\langle g_i(v_i^n), v_j^n\right\rangle\right\} + O(\Delta t^2) \tag{2.2.19}$$

and furthermore by Lemma 2.2.2

$$\begin{aligned}\left\langle v_i^n, g_j(v_j^n)\right\rangle + \left\langle v_j^n, g_i(v_i^n)\right\rangle &= \frac{1}{2\Delta t}\left(\left\langle v_i^{n+1}, v_j^n\right\rangle - \left\langle v_i^n, v_j^{n+1}\right\rangle\right) \\ &\quad + \frac{1}{2\Delta t}\left(\left\langle v_j^{n+1}, v_i^n\right\rangle - \left\langle v_j^n, v_i^{n+1}\right\rangle\right) + C_i \Delta t \\ &= C_i \Delta t. \end{aligned} \tag{2.2.20}$$

Combining (2.2.19) and (2.2.20) we find the desired result.□

We now prove that our scheme is consistent.

Theorem 2.2.3 (Consistency)

Let V_i^n denote the solution of the ith differential equation of 2.2.3 evaluated at time $t = n\Delta t$.

Then the local truncation error η_i^n satisfies

$$\eta_i^n := \frac{V_i^{n+1} - V_i^n}{\Delta t} - V_i^{**}\frac{\|V_i^n\|}{\|V_i^{**}\|} = O(\Delta t),$$

and the scheme is consistent of order Δt.

Proof By Lemma 2.2.1 we can write (2.2.8–2.2.9) as

$$V_i^{n+1} = V_i^{**} + C_i \Delta t^2.$$

If we substitute in V_i^{**} from (2.2.8) we get

$$V_i^{n+1} = V_i^n + \Delta t g_i(V_i^n) - \sum_{j=1}^{i-1} \frac{\left\langle V_i^n + \Delta t g_i(V_i^n), V_j^{n+1}\right\rangle}{\left\langle V_j^{n+1}, V_j^{n+1}\right\rangle} V_j^{n+1} + C_i \Delta t^2.$$

Applying the result of Corollary 2.2.1 and re-arranging we see that

$$\frac{V_i^{n+1} - V_i^n}{\Delta t} = g_i(V_i^n) + O(\Delta t).$$

and hence the scheme is consistent of order Δt. □

Theorem 2.2.4 *For all Δt sufficiently small the function $f_i : I\!R^{p\times i} \to I\!R^p$, $i = 1, \cdots, p$ defined in* (2.2.10) *is locally Lipschitz continuous and so for w_i, v_i in a ball of radius ρ, $B(\rho) \subset I\!R^p$, there exists a constant $C = C(\rho)$ such that*

$$\|f_i(w_i) - f_i(v_i)\| \le C\|v_i - w_i\| \quad \forall i = 1, \cdots, p.$$

Proof Follows in the same manner as Theorem 2.1.3. □

Theorem 2.2.5 (Convergence) *The scheme* (2.2.9) *is convergent of $O(\Delta t)$ to the differential equations* (2.2.2) *for all $i = 1, \cdots, p$.*

Proof

The proof of *k-stability* is completely analogous to that given in Theorem 2.1.4 and hence is omitted. Convergence follows from the "*consistency* + *k-stability*" results detailed in [117, 118] and stated in Theorem 2.0.1. □

2.3 Estimation of the Exponents

In the previous two sections we discussed schemes which allow us to numerically integrate (2.1.2) conserving the orthonormality of the initial vectors. In this section we apply those results and examine the accuracy of the numerical approximation to the Lyapunov exponent for the linear system (2.0.2). We emulate the analysis in Dieci et al [36], noting that our analysis is not restricted to the case when all exponents are calculated.

Notation

- We let μ_i denote the ith Lyapunov exponent associated with the linear system (2.0.2),

$$\mu_i = \limsup_{t\to\infty} \frac{1}{t} \int_0^t \langle V_i(s), JV_i(s)\rangle \, ds.$$

- We let $\mu_i(T)$ given by

$$\mu_i(T) = \frac{1}{T} \int_0^T \langle V_i(s), JV_i(s)\rangle \, ds$$

  be the finite time approximation to μ_i.

- We let $\widetilde{\mu}_i$

$$\widetilde{\mu}_i = \frac{1}{N\Delta t}\sum_{n=1}^{N} \Delta t \, \langle V_i^n, JV_i^n \rangle, \quad N\Delta t = T$$

be the approximation of $\mu_i(T)$ quadrature.

Lemma 2.3.1 *Given any $\epsilon > 0$, $\exists\ T \in (0,\infty)$ such that $\forall i = 1,\cdots,p$*

$$|\mu_i(t) - \mu_i| < \epsilon \qquad \forall t > T.$$

Proof See [36, Section 4].□

The key point about this lemma is that T is finite and hence we can apply standard error estimates. This leads to the following theorem.

Theorem 2.3.1 *Given any $\epsilon > 0$ let T be found from Lemma 2.3.1 with and choose $N \in I\!N$ so that $N\Delta t = T$.*

Suppose that (2.1.2) is integrated numerically on $[0,T]$ using an rth order method as described in either Section 2.1 or 2.2 and that the integral (2.1.3) is estimated by a quadrature rule of order s to get the computed approximation

$$\widetilde{\mu}_i^c = \frac{1}{N\Delta t}\sum_{n=1}^{N} \Delta t \, \langle v_i^n(s), Jv_i^n(s) \rangle .$$

Then there exists $K_i > 0$ such that we have the following bound

$$|\widetilde{\mu}_i^c - \mu_i| < C(T)\Delta t^{\min(s,r)} + \epsilon.$$

Hence for eps $= O(\Delta t^{\min(s,r)})$ we find that there exists $K_i > 0$ such that

$$|\widetilde{\mu}_i^c - \mu_i| < K_i \Delta t^{\min(s,r)}.$$

Sketch of Proof

By the quadrature and discretization errors we find

$$|\widetilde{\mu}_i^c - \mu_i(T)| \le |\widetilde{\mu}_i^c - \widetilde{\mu}_i| + |\widetilde{\mu}_i - \mu_i(T)| = C(T)\Delta t^{\min(s,r)}.$$

Thus by Lemma 2.3.1 we find

$$|\widetilde{\mu}_i^c - \mu_i| \le |\widetilde{\mu}_i^c - \mu_i(T)| + |\mu_i(T) - \mu_i| \le K_i \Delta t^{\min(s,r)}.$$

□

Note We remark that we have assumed in the above Theorem that the linear system is solved exactly.

2.4 Numerical Results

In Figure 2.4 we have computed, by the generalised scheme (2.2.8, 2.2.9) (denoted GME - Genearlised Modified Euler), the Lyapunov exponent for the Lorenz equations for $r = 45.92$, $\sigma = 16$ and $\beta = 4$. These are the same parameter values as in Figures 2.3 and used for calculating the exponents in Figure 2.2. The integral (2.1.3) was estimated by the same quadrature method as we used for Figure 2.2. Comparing Figures 2.2 and 2.4 we see that the two methods produce the same qualitative results.

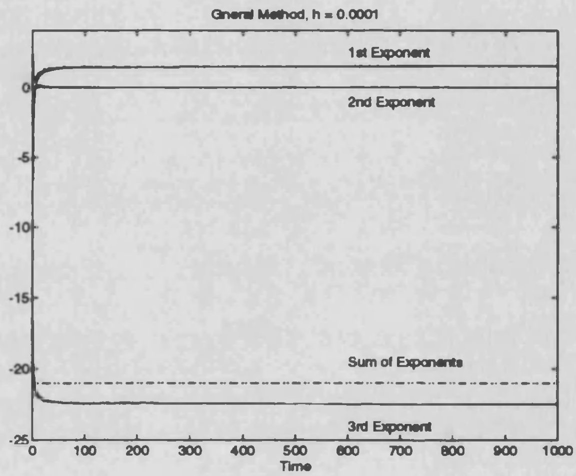

Figure 2.4: All 3 exponents for the Lorenz equations ($r = 45.92$) found by GME.

In Table 2.1 we compare our results for the Lorenz equations to those found by other authors. The method of Section 2.1.1 given by equations (2.2.8–2.2.9) is written as ME in the table, standing for "modified Euler". The scheme discussed in Section 2.2 is denoted GME in the table and stands for "generalised modified Euler". We compare our results to a statistical method (Stat), a differential method (DffMthd) and the standard method (StdMthd) :

Stat [115]: This is a statistical method for finding the Lyapunov exponents from a time series. We have taken our results from Sano and Sawada [115].

DffMthd [36]: The Lyapunov exponents were found from the differential method using a unitary integrator (ie orthogonality preserving) to integrate the equations

(2.1.2) and a composite trapezoidal rule to estimate the integral (2.1.3). These results may be found in [36].

StdMthd : This is the standard method (see section 1.2.4) for finding the Lyapunov exponents. We present results from [36] and [135].

In each (relevant) case the total integration time was taken to be $T = 1000$. For our schemes (ME and GME) we took a step size $\Delta t = 0.0001$.

Method	r	μ_1	μ_2	μ_3	$\mu_1 + \mu_2 + \mu_2$
Stat [115]	40.0	1.37 ±0.08	-0.2 ± 0.09	-15.2 ±2.1	-
StdMthd [36]	40.0	1.33961	-0.01055	-22.30930	-20.98024
DffMthd [36]	40.0	1.36006	0.00570	-22.36576	-21.00
ME	40.0	1.353239	-0.006204	-22.347250	-21.000215
StdMthd [135]	45.92	1.497	0.0	-22.458	-20.961
StdMthd [36]	45.92	1.47829	-0.01086	-22.44645	-20.97902
DffMthd [36]	45.92	1.48804	0.00452	-22.52772	-21.03516
GME	45.92	1.476634	-0.005988	-22.470644	-20.999998

Table 2.1: Comparison of different methods.

We note that our results agree very well with the other computed values and recall for these parameter values of the Lorenz equations that

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t \mathrm{Tr}(DF(U(s)))ds = -\sigma - \beta - 1 = -21.$$

Hence the sum of global Lyapunov exponents is -21. The numerical results suggest two things: first that we are computing a trajectory on the global attractor and secondly that the system is regular. Indeed if the system were non–regular it is exceedingly unlikely that we would have obtained by the above methods any reasonable numerical results (recall the non–regular system example in section 1.2.3).

In Table 2.2 we compare our results for different step sizes again integrating for a fixed time of $T = 1000$. Recall that we know the second exponent should be zero since the chaotic trajectory is stable and a zero exponent corresponds to perturbations along the trajectory. It is clear from the table that as the mesh size is refined we find a better approximation, and that the methods ME and GME yield the same results to order 10^{-3}.

It is of interest that the sum of the exponents is estimated far more accurately than the individual exponents (for example $\Delta t = 0.01$). This illustrates that the sum of exponents should not be used as a stopping criteria in codes.

It should be remembered that there are two limiting processes at work: there is the limit as $\Delta t \to 0$ and the limit as $T \to \infty$. Figures 2.5 illustrate the two limiting processes. The x co-ordinate of each plot is T and the y co-ordinate of each plot is $\log(\Delta t)$. If we fix a point on the $\log(\Delta t)$ axis and look along the T axis we are looking at convergence for a fixed Δt in time. If we fix T and look along the $\log(\Delta t)$ axis we are looking at convergence as $\Delta t \to 0$. On these plots the point $(50, -10, \bullet)$ is the most accurate approximation.

Method	r	μ_1	μ_2	μ_3	$\mu_1 + \mu_2 + \mu_2$	Δt
ME	40.0	0.945100	-1.096546	-20.848659	-21.000105	0.01
ME	40.0	1.308091	-0.102461	-22.205609	-20.999979	0.001
ME	40.0	1.353239	-0.006204	-22.347250	-21.0000215	0.0001
GME	40.0	0.431585	-0.671833	-20.759647	-20.999895	0.01
GME	40.0	1.265091	-0.059433	-22.205648	-20.99999	0.001
GME	40.0	1.352325	-0.005025	-22.347299	-20.9999999	0.0001
ME	45.92	1.206253	-1.277307	-20.928842	-20.996896	0.01
ME	45.92	1.429261	-0.119475	-22.309775	-20.999989	0.001
ME	45.92	1.478099	-0.007642	-22.470456	-20.999999	0.0001
GME	45.92	0.511754	-0.773511	-20.738033	-20.99979	0.01
GME	45.92	1.386088	-0.074614	-22.311453	-20.999979	0.001
GME	45.92	1.476634	-0.005988	-22.470644	-20.999998	0.0001

Table 2.2: Comparaison of the schemes ME and GME.

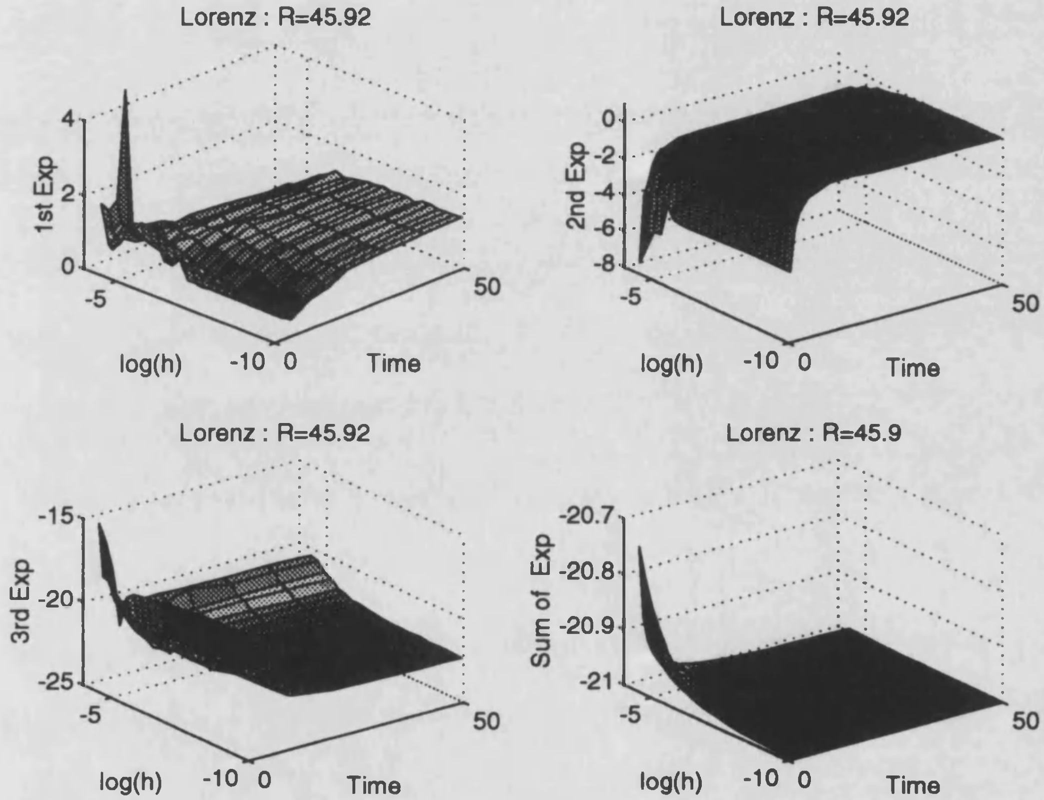

Figure 2.5: Exponents and sum of exponents for Lorenz equations ($r = 45.92$).

Chapter 3

The Complex Ginzburg–Landau Equation

Notation : Let $z \in \mathbb{C}$, then

- $\overline{z}$ will denote the complex conjugate of z,
- $|z|$ will denote the modulus of z.

3.1 The Equation

The complex Ginzburg–Landau equation models the evolution of the amplitude of perturbations to steady state solutions at the onset of instability. We shall consider the Ginzburg–Landau equation with cubic non–linearity on an open bounded subset Ω of $\mathbb{R}^p$, $p = 1, 2$,

$$\begin{aligned} U_t &= RU - (1 + i\nu)A_0 U - (1 + i\mu)|U|^2 U \\ U^0 &= U(0) \end{aligned} \tag{3.1.1}$$

where A_0 is the linear operator given by

$$A_0 := -\Delta \text{ with domain } D(A_0) := \left\{ u \in L^2_{\text{per}} : |A_0 u|_{L^2} < \infty \right\}; \tag{3.1.2}$$

and R, ν and μ are all real parameters. We have that $U(x,t) : \Omega \times \mathbb{R}^+ \to \mathbb{C}$ and assume that

- when $p = 1$, $\Omega = [0, 1]$;

- when $p = 2$, $\Omega = [0, 1] \times [0, 1]$.

We supplement equation (3.1.1) with periodic boundary conditions on Ω.

The equation arises in many areas of physics as it models many fundamental phenomena. In fluid dynamics it is sometimes referred to as the Stuart–Stewartson equation and is found, for example, in the study of Poiseuille flow, Rayleigh–Bénard convection and Taylor–Couette flow [123, 84, 41, 110, 111, 39]. The equation is also used to model the transition to turbulence in chemical mediums [75, 76]. The equation derives the name used here from the study of super–conductivity where it models the phase transition of the material from superconducting phase to a non–superconducting phase [84, 41, 110, 111, 39].

3.2 An Abstract Framework

Let us consider the abstract evolution equation

$$u_t + Au = F(u) \tag{3.2.3}$$

$$u^0 = u(0)$$

in a Hilbert space X with inner–product $(\bullet, \bullet)$ and norm $\| \bullet \|_X^2 = (\bullet, \bullet)$.

The aim is to state conditions under which we may define arbitrary positive powers of the linear operator A and hence define certain function spaces. To start we recall some fundamental properties and definitions primarily from Henry [68] and Pazy [104].

Definition 3.2.1 For a linear operator A in a Hilbert space X, $\lambda \in \mathbb{C}$ is said to be an *eigenvalue* of A if and only if there exists a non–trivial $\Psi \in X$ such that

$$A\Psi = \lambda\Psi.$$

In which case $\Psi \in X$ is termed an *eigenfunction* of A. The *resolvent set*, $\rho(A)$, of A is defined to be the set

$$\rho(A) := \{\lambda \in \mathbb{C} \, : (\lambda I - A) \text{ is invertible}\}$$

and we refer to the operator $R(\lambda : A) := (\lambda I - A)^{-1}$ for $\lambda \in \rho(A)$, as the *resolvent* of A. The *spectrum* of A, $\sigma(A)$, is defined by $\sigma(A) := \mathbb{C} \setminus \rho(A)$.

Definition 3.2.2 (Sectorial Operator) *[68, 1.3.1]*

The linear operator A is said to be *sectorial* if the following hold

a) A is closed, i.e. for $\{x_n\} \in D(A)$ *s.t.* $x_n \to x$ in $D(A)$ and $Ax_n \to y$ in X we have $Ax = y$,

b) A is densely defined, i.e. $\overline{D(A)} = X$,

c) There exists $\theta \in (0, \pi/2)$, $M > 1$, and $a \in \mathbb{R}$ such that if $\Sigma_{a,\theta}$ is the sector given by

$$\Sigma_{a,\theta} := \{\lambda \in \mathbb{C} \setminus \{a\} : |\arg(\lambda - a)| < \theta\},$$

then

$$\sigma(A) \subset \Sigma_{a,\theta} \tag{3.2.4}$$

and

$$\|(\lambda I - A)^{-1}\| \leq \frac{M}{|\lambda - a|} \quad \forall\, \lambda \in \mathbb{C} \setminus \Sigma_{a,\theta}. \tag{3.2.5}$$

Definition 3.2.3 (Analytic Semigroup) *[104, 2.5.1]*

Let $\Delta = \{z \in \mathbb{C} : \theta_1 < \arg(z) < \theta_2, \theta_1 < 0 < \theta_2\}$ and for $z \in \Delta$ let $L(z) : X \to X$ be a bounded linear operator. The family $L(z), z \in \Delta$, is said to be an *analytic semi–group* in Δ if

a) $L(0) = I$ and $\lim_{\substack{z \to 0 \\ z \in \Delta}} L(z)x = x \;\forall\, x \in X$,

b) $L(z_1 + z_2) = L(z_1)L(z_2) \;\forall z_1, z_2 \in \Delta$,

c) $z \to L(z)$ is analytic in Δ.

The linear operator $-A$ is said to be the *infinitesimal generator* of the semi–group $\{L(z)\}_{z \in \Delta}$ if

$$-Ax = \lim_{z \to 0+} \frac{L(z)x - x}{z};$$

and the domain of $-A$ ($D(-A) = D(A)$) consists of all $x \in X$ such that the limit exists.

We now state the equivalence of these two concepts:

Theorem 3.2.1 *A is a sectorial operator if and only if $-A$ is the infinitesimal generator of an analytic semi–group.*

Proof See, for example, either [104, 2.5.2] or [68, 1.3.4].□

The following theorem gives a sufficient condition for A to be a sectorial operator.

Theorem 3.2.2 *If the linear operator A is*

- *self-adjoint, i.e. $\forall u, v \in D(A), (Au, v) = (u, Av)$;*
- *coercive, i.e. $\forall u \in D(A),\ \exists \alpha \geq 0\ \ s.t. \langle Au, u \rangle\ \geq \alpha \|u\|_X^2$;*
- *densely defined;*

then A is a sectorial operator.

Proof See [68].

We now define arbitrary positive powers of A.

Theorem 3.2.3 *Let A be a sectorial operator then for all $\alpha > 0$ we have that the integral*

$$\frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-At} dt,$$

converges in the uniform operator topology and hence $A^{-\alpha}$ defined by

$$A^{-\alpha} := \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-At} dt,$$

is well defined. Furthermore we may define A^α, $\forall \alpha > 0$ by

$$A^\alpha = \left(A^{-\alpha}\right)^{-1}.$$

Proof See for example [68, 1.4.1],[104, 2.6] or [51].□

Notes:

- Theorem 3.2.3 allows us to define for all $\alpha > 0$ the Banach space

  $$X^\alpha := D\left(A^\alpha\right) = \{u \in X : \|A^\alpha u\|_X < \infty\}$$

  with associated norm $\| \bullet \|_{X^\alpha}$ given by

  $$\| \bullet \|_{X^\alpha} := \|A^\alpha u\|_X.$$

• For $\alpha = 0$ we simply define $A^0 := I$ and $X^0 = X$.

Theorem 3.2.4 *If A is a sectorial operator in a Hilbert space X, then X^α is a Hilbert space for the norm $\| \bullet \|_{X^\alpha} = (A^{\alpha/2}\bullet, A^{\alpha/2}\bullet)^{1/2}$ and for $\alpha > \beta \geq 0$, X^α is a dense subspace of X^β with continuous inclusion. Furthermore if A has a compact resolvent then the inclusion $X^\alpha \subset X^\beta$ is compact.*

Proof See Theorem 1.4.8. of Henry[68].

• If we further suppose that A is self–adjoint and hence there exists a complete orthonormal set of eigenfunctions $\{\Psi_k\}_k$ and eigenvalues $\{\Lambda_k\}_k$ for A then we have the spectral representation

$$(u, v)_{X^\alpha} = \sum_k \Lambda_k^{2\alpha}(u, \Psi_k)(v, \Psi_k). \tag{3.2.6}$$

Thus we may characterise $D(A^\alpha)$ by

$$D(A^\alpha) = \left\{ u \in X : \sum_k \Lambda_k^{2\alpha}(u, \Psi_k)^2 < \infty \right\}. \tag{3.2.7}$$

For further details see [141, 109] or [68].

We now return attention to the initial value problem given by (3.2.3). We seek conditions under which there exists a unique solution to (3.2.3); but first we need to define what we mean by a solution.

Definition 3.2.4 A function $u : [0,T) \to X$ is said to be a (classical) *solution* to (3.2.3) on $[0,T)$ if u is continuous on $[0,T)$ and continuously differentiable on $(0,T)$, $u(t) \in D(A)$ on $(0,T)$ and (3.2.3) is satisfied on $[0,T)$.

It is a classical result that if (3.2.3) has a solution then it satisfies the integral equation

$$u(t) = L(t)u^0 + \int_0^t L(t-s)F(u(s))\, ds. \tag{3.2.8}$$

This may be found in many works and is also contained in both [68] or [104]. It leads to the definition of a weak solution.

Definition 3.2.5 A continuous solution u of the integral equation (3.2.8) is defined to be a *weak solution* of (3.2.3).

Theorem 3.2.5 *Let $F : X \to X$ be uniformly Lipschitz continuous with constant M on X. If $-A$ is the infinitesimal generator of a continuous semi-group then for every $u^0 \in X$ the initial value problem* (3.2.3) *has a unique weak solution $u \in C([0,T] : X)$. Moreover the mapping $S(t)u^0 = u$ is Lipschitz continuous from X into $C([0,T], X)$.*

Proof A classical result, see for example Theorem 6.1.2 of [104]. □

Theorem 3.2.6 *Let $F : X \to X$ be uniformly Lipschitz continuous with constant M on $D(A)$. If $u^0 \in D(A)$ then for each t_0, $0 < t_0 < T$ the initial value problem* (3.2.3) *has a unique classical solution on $[t_0, T]$ where at $t = t_0$ the initial condition is satisfied in a weak sense.*

Proof See Theorem 6.1.7. of Pazy [104] □

Assumption $\mathcal{F}$:

Let E be an open subset of X^α. The function f is said to satisfy *Assumption $\mathcal{F}$* on E if, for every $x \in E$, $\exists$ a neighbourhood $W \subset E$ and constant $M \geq 0$ such that

$$\|f(x_1) - f(x_2)\| \leq M\|x_1 - x_2\|_{X^\alpha}$$

for all $x_1, x_2 \in W$.

Theorem 3.2.7 *Suppose that $-A$ is the infinitesimal generator of a bounded analytic semi-group $\{L(t)\}_{t\geq 0}$ and that $0 \in \rho(-A)$. Furthermore suppose $\exists M, \alpha > 0$ such that the function F satisfies assumption $\mathcal{F}$ on $E \subset X^\alpha$. Then for every $U^0 \in E$ the initial value problem* (3.2.3) *has a unique local solution $u \in C([0,t_1) : X) \bigcup C^1((0,t_1) : X)$ with $t_1 = t_1(U^0)$.*

Proof See Theorem 6.3.1 of Pazy [104].□

Theorem 3.2.8 *Suppose that A and F are as in Theorem 3.2.7 and, in addition, that for every closed bounded set $B \subset E$, $F(B)$ is bounded in X. If u is a solution of* (3.2.3) *on $(0, t_1)$ and t_1 is maximal, so there is no solution of* (3.2.3) *on $(0, t_2)$ if $t_2 > t_1$, then either $t_1 = \infty$ or else there exists a sequence $t_n \to t_1$ as $n \to \infty$ such that $u(t_n) \to \partial E$. (If E is unbounded the point at infinity is included in ∂E).*

Proof See Theorem 3.3.4 of Henry [68]. □

Consider the heat equation

$$u_t + Au = 0, \quad u^0 = u(0), \quad \text{where } A := -\Delta, \ , D(A) = \{v \in L^2 : |Av|_{L_2} < \infty\}$$

for which it is well known that solutions become instantaneously smooth, see for example [81]. This is due to the smoothing action of the Laplacian operator. In this general setting it can be proved that if A is sectorial then A has the same kind of smoothing property. This is stated in the next Theorem.

Theorem 3.2.9 (Smoothing) *Let $-A$ be the infinitesimal generator of an analytic semi-group $L(t)$. If $0 \in \rho(A)$ then,*

(a) $L(t) : X \to D(A^\alpha)$ for every $t > 0$ and $\alpha > 0$.

(b) For every $x \in D(A^\alpha)$ we have $L(t)A^\alpha x = A^\alpha L(t)x$.

(c) For every $t > 0$ the operator $A^\alpha L(t)$ is bounded and there exists a constant $\delta > 0$ such that

$$\|A^\alpha L(t)\| \leq Mt^{-\alpha}e^{-\delta t}.$$

Proof See Pazy [104], Theorem 2.6.13. □

3.3 The Mathematical Setting

The purpose of this section is to introduce the functional setting for the Ginzburg–Landau equation. For other overviews see [128] or [102]. The spaces which are defined below will be used without reference throughout the rest of this thesis.

We consider (3.1.1) as an ordinary differential equation in $X = L^2 := L^2_{per}(\Omega)$ the Lebesgue measure space, with inner–product $< \bullet, \bullet >$ given by

$$< u, v >= \int_\Omega u\bar{v} \, dx. \tag{3.3.9}$$

The inner–product (3.3.9) induces the standard L^2 norm $| \bullet |^2_{L^2} =< \bullet, \bullet >$.

L^p Spaces:

We define the Lebesgue measure spaces $L^p = L^p_{per}(\Omega)$ and their norms in the standard manner

$$|v|_{L^p} = \begin{cases} \left(\int_\Omega |v|^p dx\right)^{1/p} & 1 \leq p < \infty \\ \operatorname{ess\,sup} |v| \,, & p = \infty. \end{cases} \tag{3.3.10}$$

See for example [1] or [109].

A Sectorial Operator:

We now construct function spaces for the Ginzburg–Landau equation using powers of a linear operator. In order for these powers to be well defined by Theorem 3.2.3 it is sufficient to consider a sectorial operator. Since the operator A_0 defined by (3.1.2) is not sectorial; we introduce the linear operator $\widetilde{A}_0$ defined by

$$\widetilde{A}_0 := I + A_0, \tag{3.3.11}$$

with domain, $D(\widetilde{A}_0) = D(A_0)$.

Theorem 3.3.1

The linear operator A_0 defined by (3.1.2) *has eigenvalues* $\{\Lambda_k\}_k$, $k \in \mathbb{Z}$, *where*

$$\Lambda_k = 4k^2\pi^2, \quad k \in \mathbb{Z}.$$

The linear operator $\widetilde{A}_0$ defined by (3.3.11) *has eigenvalues* $\left\{\widetilde{\Lambda}_k\right\}_k$, $k \in \mathbb{Z}$ *where*

$$\widetilde{\Lambda}_k = 1 + 4k^2\pi^2 \quad k \in \mathbb{Z}. \tag{3.3.12}$$

The corresponding eigenfunctions $\{\Psi_k\}_{-\infty}^{\infty}$, *identical for A_0 and $\widetilde{A}_0$, form a complete orthonormal set contained in L^2. In dimension $p = 1$ the eigenfunctions are given by*

$$\Psi_k = e^{2\pi i k x}, \quad k \in \mathbb{Z}. \tag{3.3.13}$$

Proof See, for example, [68, 81] or [109].□

Theorem 3.3.2 *The linear operator $\widetilde{A}_0$ defined by* (3.3.11) *is a sectorial operator.*

Proof

First we prove that $\widetilde{A}_0$ is closed and densely defined. By definition

$$|\widetilde{A}_0 u|_{L^2}^2 = \int_\Omega (I-\Delta)u(I-\Delta)\overline{u}\,dx, \quad u \in D(\widetilde{A}_0)$$

which by Green's theorem and the periodic boundary conditions becomes

$$|\widetilde{A}_0 u|_{L^2}^2 = \int_\Omega (|u|^2 + 2|\nabla u|^2 + |\Delta u|^2)\,dx. \tag{3.3.14}$$

Hence $D(\widetilde{A}_0)$ is norm equivalent to the Sobolev space $H^2_{per}(\Omega)$ defined by distributional derivatives (see for example [1, 68, 81, 104, 128]). Now by the Rellich–Kondrachov Theorem [1, Chapter 5] we have that $H^2_{per}(\Omega)$ is compactly embedded in L^2_{per} from which it follows that $\widetilde{A}_0$ is closed and densely defined.

It remains to prove part **c)** of Definition 3.2.2 of a sectorial operator. By Theorem 3.3.1, $\sigma(\widetilde{A}_0)$, consists of a set of strictly positive discrete real points and hence (3.2.4) is proved for $a = 0$ and any $\theta \in (0, \pi/2)$. In order to prove (3.2.5) note that for $\lambda \in \mathbb{C} \setminus \Sigma_{0,\theta}$, the operator $(\lambda I - \widetilde{A}_0)^{-1}$ exists. Now consider

$$\lambda\eta - \widetilde{A}_0\eta = f, \tag{3.3.15}$$

and note that by definition and (3.3.15)

$$|(\lambda I - \widetilde{A}_0)^{-1}|_{L^2} = \sup_{f\in L^2} \frac{|(\lambda I - \widetilde{A}_0)^{-1} f|_{L^2}}{|f|_{L^2}} = \sup_{f\in L^2} \frac{|\eta|_{L^2}}{|f|_{L^2}}.$$

By Theorem 3.3.1 we can expand η and f as Fourier series

$$\eta = \sum_{k=-\infty}^{\infty} a_k \Psi_k, \quad f = \sum_{k=-\infty}^{\infty} f_k \Psi_k,$$

to get from (3.3.15)

$$(\lambda - \Lambda_k)a_k = f_k.$$

Since $\lambda \in \mathbb{C} \setminus \Sigma_{0,\theta}$ we have that

$$\frac{|\eta|_{L^2}^2}{|f|_{L^2}^2} = \frac{\sum_{k=-\infty}^{\infty} \frac{|f_k|^2}{|\lambda-\Lambda_k|^2}}{\sum_{k=-\infty}^{\infty} |f_k|^2} \le \max_k \frac{1}{\lambda - \Lambda_k}.$$

Hence,

$$|(\lambda I - \widetilde{A}_0)^{-1}|_{L^2} = \sup_{f\in L^2} \frac{|\eta|_{L^2}}{|f|_{L^2}} \le \max_k \frac{1}{|\lambda - \Lambda_k|} \le \frac{1}{\sin(\theta)|\lambda|},$$

and we have proved that $\widetilde{A}_0$ is a sectorial operator.□

Since the operator $\widetilde{A}_0$ is sectorial by Theorem 3.2.3 powers of $\widetilde{A}_0$ are well defined. Our first application of this fact will be to define the Sobolev spaces.

Sobolev Spaces : H^{2s}

Let $u \in L^2$; then, by Theorem 3.3.1, u may be defined by a Fourier series :

$$u = \sum_{k=-\infty}^{\infty} a_k \Psi_k, \; a_k =< u, \Psi_k > .$$

The **Sobolev space** $\{H^{2s}, \| \bullet \|_{H^{2s}}\}$ is then defined as

$$H^{2s} := D(\widetilde{A}_0^s) = \left\{ u \in L^2 : |\widetilde{A}_0^s u|_{L^2} < \infty \right\}; \tag{3.3.16}$$

where,

$$\|u\|_{H^{2s}}^2 := |\widetilde{A}_0^s u|_{L^2}^2 = \sum_{k=-\infty}^{\infty} \Lambda_k^{2s} |a_k|^2. \tag{3.3.17}$$

For example

1. when $s = 0$ the space H^0 is exactly the space L^2;

2. when $s = \frac{1}{2}$ the space $H^1 = D(\widetilde{A}_0^{1/2})$ with norm,

$$\|u\|_{H^1}^2 = \sum_{k=-\infty}^{\infty} \widetilde{\Lambda}_k < u, \Psi_k >^2 .$$

Theorem 3.3.3

The spaces H^{2s} defined by (3.3.16,3.3.17) are equivalent to the Sobolev spaces defined through distributional derivatives.

Proof See for example either [1] or [71].□

In the analysis which follows we shall frequently use the Sobolev space H^1 and therefore we note the particular case of Theorem 3.3.3 with $s = 1/2$.

Recall the standard definition of the H^1 norm (see for example any of [1, 51, 68, 81, 104, 128, 141]). We define the inner–product in H^1 by

$$< u, v >_A =< \nabla u, \nabla v >= \int_\Omega \nabla u \nabla \overline{v} \, dx \tag{3.3.18}$$

which induces the H^1 semi–norm

$$\|u\|_1^2 =< u, u >_A = \int_\Omega |\nabla u|^2 dx. \tag{3.3.19}$$

The full H^1 norm defined by distributional derivatives is given by

$$\|u\|_{H^1} := \left\{|u|_{L^2}^2 + \|u\|_1^2\right\}^{1/2}. \tag{3.3.20}$$

Theorem 3.3.3 states that the norm (3.3.20) and (3.3.17) with $s = \frac{1}{2}$ are equivalent.

Proof of Theorem 3.3.3 with $s = 1/2$.

By Definition (3.3.20) of the H^1 norm and the Fourier expansion for u

$$\begin{aligned}
\|u\|_{H^1}^2 &= |u|_{L^2}^2 + \|u\|_1^2 = \int_\Omega |u|^2 dx + \int_\Omega |\nabla u|^2 dx \\
&= \sum_{k=-\infty}^{\infty} |a_k|^2 + \sum_{k=-\infty}^{\infty} 4k^2\pi^2 |a_k|^2 = \sum_{k=-\infty}^{\infty} (1 + 4k^2\pi^2)|a_k|^2 \\
&= \sum_{k=-\infty}^{\infty} \widetilde{\Lambda}_k |a_k|^2,
\end{aligned}$$

which is exactly the H^1 norm defined by (3.3.17) with $s = \frac{1}{2}$.□

Gevrey Class

We now use powers of the linear operator $\widetilde{A}_0$ to define a special class of Gevrey spaces. We refer the reader to [106, p12] or [32, p240] for the more general definitions and to [106, 54, 32] for related results and further references. The definition of Gevrey class and regularity which we give here is the same as that used by [49, 40] and [42]. First we note that functions of unbounded operators (and so functions of the operator $\widetilde{A}_0^s$, $s > 0$) are defined in the work Dunford and Taylor [43].

Let $u \in L^2$ have Fourier expansion

$$u = \sum_{k=-\infty}^{\infty} a_k \Psi_k.$$

Then we define the *Gevrey space* $G_{\tau,s}$ by

$$G_{\tau,s} := D\left(\widetilde{A}_0^s e^{\tau \widetilde{A}_0^s}\right) = \left\{u \in L^2 : |\widetilde{A}_0^s e^{\tau \widetilde{A}_0^s}|_{L^2}^2 < \infty\right\} \tag{3.3.21}$$

with norm denoted $\|u\|_{G_{\tau,s}}$,

$$\|u\|_{G_{\tau,s}}^2 = |\widetilde{A}_0^s e^{\tau \widetilde{A}_0^s} u|_{L^2}^2 = \sum_{k=-\infty}^{\infty} \widetilde{\Lambda}_k^{2s} e^{2\tau \widetilde{\Lambda}_k^s} |a_k|^2.$$

Note : We only consider the case when $s = \frac{1}{2}$ and $\tau > 0$. Henceforward we shall use G_τ to denote $G_{\tau,s}$.

As is standard practice, we shall call τ the order of the Gevrey regularity of G_τ and we shall speak of $u \in G_\tau$ being of Gevrey class of regularity τ. The following Theorem relates Gevrey classes of differing regularity.

Lemma 3.3.1 *For $\tau > \sigma > 0$ we have that $G_\tau \subset G_\sigma$.*

Proof Immediate from the definition of the Gevrey spaces. □

From the definition of the Gevrey spaces it is clear that if $u \in G_\tau$ then the Fourier coefficients of u decay exponentially in k and hence u is a smooth analytic function. The next theorem states exactly how smooth it is.

Theorem 3.3.4 *For all $\alpha \in I\!\!R^+$, and for all $\tau > 0$ the following inclusion is true :*

$$G_\tau \subset H^{2\alpha}.$$

Proof Let $u \in G_\tau$, then $\|u\|_{G_\tau} \leq C < \infty$. Now,

$$\begin{aligned} |\widetilde{A}_0^\alpha u|_{L^2}^2 &= \sum_{k=-\infty}^{\infty} \widetilde{\Lambda}_k^{2\alpha} |a_k|^2 \\ &= \sum_{k=-\infty}^{\infty} \widetilde{\Lambda}_k e^{\tau \widetilde{\Lambda}_k^{1/2}} |a_k|^2 \left\{ \widetilde{\Lambda}_k^{2\alpha-1} e^{-\tau \widetilde{\Lambda}_k^{1/2}} \right\}. \end{aligned}$$

All that remains is to prove that $\widetilde{\Lambda}_k^{2\alpha-1} e^{-\tau \widetilde{\Lambda}_k^{1/2}}$ is bounded uniformly, independently of k. Let $f(x) = x^\beta e^{-2\tau x^{1/2}}$. Then,

$$f'(x) = x^{\beta-1} e^{-2\tau x^{1/2}} \left\{ \beta - \tau x^{1/2} \right\}.$$

Thus we have a global maximum at $x^{1/2} = \beta/\tau$ at which

$$f(x) = (\beta/\tau)^{2\beta} e^{-2\beta},$$

and is independent of x. Hence the theorem is proved.□

Theorem 3.3.5 *If $u \in G_\tau$ for some $\tau > 0$ then u is real analytic and hence $u \in C^\infty$.*

Proof The proof that u is both real analytic and C^∞ follow from John [81, p.64-65].
□

3.4 Return of the Ginzburg–Landau Equation

Notation :

We shall employ the following notation to denote balls in the spaces defined in Section 3.3. The ball of radius ρ centre 0 in L^2 $(= H^0)$ is defined by

$$\mathcal{B}_0(\rho) := \{v \in L^2 : |v|_{L^2} \leq \rho\};$$

the ball in H^1 centre 0 radius ρ by

$$\mathcal{B}_1(\rho) := \{v \in H^1 : \|v\|_{H^1} \leq \rho\};$$

and finally the ball in G_τ centre 0 radius ρ by

$$\mathcal{B}_{G_\tau}(\rho) := \{v \in G_\tau : \|v\|_{G_\tau} \leq \rho\}.$$

We now apply the results of section 3.2 and 3.3 to the complex Ginzburg–Landau equation (3.1.1). To do that we reformulate equation (3.1.1) using the operator $\widetilde{A}_0$ defined in (3.3.11)

$$U_t + (1+i\nu)\widetilde{A}_0 U = F_0(U), \tag{3.4.22}$$

$$U^0 = U(0),$$

where

$$F_0(V) = \widetilde{R}V - (1+i\mu)|V|^2 V \tag{3.4.23}$$

and $\widetilde{R} = R + (1+i\nu)$.

We show that the following properties **C1-C4** hold.

C1 The family of solution operators forms a continuous semi–group in L^2.

C2 There is an absorbing set $\mathcal{B}_0(\rho_0)$.

C3 There is an absorbing set in $\mathcal{B}_1(\rho_1)$.

C4 There is a global attractor $\mathcal{A}$ in L^2.

We apply the theory developed in Section 3.2 and the results of Section 3.3 to prove existence and uniqueness of solutions to (3.4.22) and hence the existence of a non–linear semi-group. First however we prove some Lipschitz inequalities for the function F_0 defined by (3.4.23).

Lemma 3.4.1 *For all $u, v \in \mathbb{C}$ the following inequality holds*

$$\big||u|^2u - |v|^2v\big| \leq \frac{3}{2}(|u|^2 + |v|^2)|u - v|.$$

Proof The result follows from simple algebra :

$$\begin{aligned}
\big||u|^2u - |v|^2v\big| &= \left|u^2\overline{u} - v^2\overline{v}\right| \\
&= \left|(u-v)|u|^2 + v|u|^2 - v^2\overline{v}\right| \\
&= \left|(u-v)|u|^2 + (\overline{u} - \overline{v})vu + |v|^2(u-v)\right| \\
&\leq (|u|^2 + |uv| + |v|^2)|u - v| \\
&\leq \frac{3}{2}(|u|^2 + |v|^2)|u - v|.\square
\end{aligned}$$

Lemma 3.4.2 *For all $u, v \in \mathbb{C}$ the function F_0 defined by* (3.4.23) *satisfies*

$$|F_0(u) - F_0(v)| \leq M\left\{1 + |u|^2 + |v|^2\right\}|u - v|,$$

where $M := \max\left\{R, 3/2(1+\mu^2)^{1/2}\right\}$.

Proof Note that by the triangle inequality

$$\begin{aligned}
|F_0(u) - F_0(v)| &= \left|R(u-v) - (1+i\mu)\left(|u|^2u - |v|^2v\right)\right| \\
&\leq R|u - v| + (1+\mu^2)^{1/2}\left||u|^2u - |v|^2v\right|,
\end{aligned}$$

which by Lemma 3.4.1 becomes

$$|F_0(u) - F_0(v)| \leq R|u - v| + \frac{3}{2}(1+\mu^2)^{1/2}\left(|u|^2 + |v|^2\right)|u - v|,$$

and the result is immediate.□

Lemma 3.4.3 *The function F_0 defined by* (3.4.23) *is Lipschitz continuous from H^1 to L^2.*

Proof Let $u, v \in H^1$, then by Lemma 3.4.2 and Hölder's inequality we have that

$$\begin{aligned}|F_0(u) - F_0(v)|_{L^2} &= \left\{\int_\Omega \left|\tilde{R}(u-v) + (1+i\mu)\left(|u|^2u - |v|^2v\right)\right|^2 dx\right\}^{1/2} \\ &\leq M\left\{\int_\Omega [1+|u|^2+|v|^2]^q\, dx\right\}^{1/q}\left\{\int_\Omega |u-v|^p dx\right\}^{1/p} \\ &= M\left\{\int_\Omega [1+|u|^2+|v|^2]^q\, dx\right\}^{1/q} |u-v|_{L^p},\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$, $p, q \geq 2$.

By Minkowski's inequality we see that

$$|F_0(u) - F_0(v)|_{L^2} \leq M\left\{1 + \left||u|^2\right|_{L^q} + \left||v|^2\right|_{L^q}\right\}|u-v|_{L^p},$$

and noting that

$$\left||v|^2\right|_{L^q} = \left\{\int_\Omega |u|^{2q} dx\right\}^{1/q} = \left\{\int_\Omega |u|^{q'} dx\right\}^{2/q'} = |u|^2_{L^{q'}},$$

we find

$$|F_0(u) - F_0(v)|_{L^2} \leq M\left\{1 + |u|^2_{L^q} + |v|^2_{L^q}\right\}|u-v|_{L^p},$$

where $\frac{1}{p}+\frac{2}{q}=\frac{1}{2}$, $p \geq 2, q \geq 4$.

Now by the Sobolev embedding theorem (see for example [1, 81]) we have that there exists a constant C such that

$$|u|_{L^q} \leq C|u|_{H^1}\ \forall q \geq 1.$$

Hence there exists M_1 such that

$$|F_0(u) - F_0(v)|_{L^2} \leq M_1|u-v|_{L^p},\ \forall p \geq 2$$

and so

$$|F_0(u) - F_0(v)|_{L^2} \leq M_1|u-v|_{L^2}.$$

□

Theorem 3.4.1 *Given $U^0 \in H^1$ there exists $t_{\max} \leq \infty$ such that there exists a unique solution to* (3.4.22) *on* $[0, t_{max})$.

If we further assume **C3**, *so there exists a uniform bound on the H^1 norm, then there exists a unique global solution to* (3.4.22) *and the mapping $U^0 \to U(t)$ is continuous and bounded.*

Proof Since $\widetilde{A}_0$ is a sectorial operator and by Lemma 3.4.3 F_0 is Lipschitz continuous from H^1 to L^2, all the conditions hold for Theorem 3.2.7. Global existence assuming a uniform H^1 bound follows immediately from Theorem 3.2.7 and Theorem 3.2.8.□

The previous Theorem gives existence of a semi–group from H^1 to H^1. We now state a stronger existence result.

Theorem 3.4.2 (C1 : Existence and Uniqueness)

Given $U^0 \in L^2$ there exists a unique solution

$$U \in C([0,T]; L^2) \bigcap L^2(0,T,H^1) \ \forall T < \infty,$$

and the mapping $U^0 \rightarrow U(t)$ is continuous and bounded from L^2 into itself $\forall t > 0$.

Proof This may be proved using a Faedo–Galerkin approach and details may be found in [128].□

3.4.1 A Global Attractor

This section commences with the statement of three useful Gronwall lemmas which will be referred to throughout this thesis. We then outline the analysis presented in Temam [128] of **C2**, **C3** and **C4**. This analysis is valid in both 1 and 2 dimensions ($p = 1, 2$).

We shall then summarise two other results which are specific to $p = 1$ dimension. The first of these results was suggested to us by Dr E. Süli [126] and simply utilises the 1 dimensional Sobolev embedding estimates (see any of [1, 51, 68, 81, 104, 128, 141]). The other is due to Doering et al [41] who treat the non–linear term more delicately and obtain results which are dependent on the parameters.

Lemma 3.4.4 (Gronwall Lemma)

Suppose there exists constants $K_1, K_2, \alpha, \beta > 0$ and $T > 0$ such that $y(t)$ satisfies the following inequality

$$0 \leq y(t) \leq K_1 t^{-1+\alpha} + k_2 \int_0^t (t-s)^{-1+\beta} y(s) ds \ \forall \ t \in (0,T].$$

Then,

$$y(t) \leq C(K_2, T, \tau, \alpha, \beta) K_1 t^{-1+\alpha} \ \forall \ t \in (0,T].$$

Proof See for example [68, Lemmas 6.3 and 7.1]. □

Lemma 3.4.5 (Standard Gronwall Lemma)

Let $g(s), h(s), y(s)$ be three positive locally integrable functions on (t_0, ∞) such that y' is locally integrable on $(0, \infty)$ and

$$\frac{d}{dt}y \leq gy + h \quad \forall\ t \geq t_0.$$

Then,

$$y(t) \leq y(t_0) \exp\left(\int_{t_0}^{t} g(\tau)d\tau\right) + \int_{t_0}^{t} h(s) \exp\left(-\int_{t}^{s} g(\tau)d\tau\right) ds \ \forall t \geq t_0.$$

Proof See for example [128, p88]. □

Lemma 3.4.6 (Uniform Gronwall Lemma)

Let $g(s), h(s), y(s)$ be three positive locally integrable functions on (t_0, ∞) such that y' is locally integrable on $(0, \infty)$ and

$$\frac{d}{dt}y \leq gy + h \quad \forall\ t \geq t_0,$$

and

$$\int_{t}^{t+r} g(s)ds \leq a_1(r), \quad \int_{t}^{t+r} h(s)ds \leq a_2(r), \quad \int_{t}^{t+r} y(s)ds \leq a_3(r) \ \forall t \geq t_0$$

for arbitrary $r > 0$. Then,

$$y(t+r) \leq \left(\frac{a_3(r)}{r} + a_2(r)\right) \exp(a_1(r)) \ \forall t \geq t_0.$$

Proof See for example [128, p89]. □

ABSORBING SET IN L_2

Theorem 3.4.3 (C2) *There exists a constant $\rho_0 = \rho_0(R) > 0$ such that the ball $\mathcal{B}_0(\rho_0)$ is absorbing and positively invariant for the semi-group $\{S(t)\}_{t \geq 0}$.*

That is, for any $B \subset \mathcal{B}_0(\rho)$, $\exists\ t_0 = t_0(\rho, \rho_0)$ such that for all $t > t_0$

$$S(t)B \subset \mathcal{B}_0(\rho_0).$$

Proof The proof is via an energy argument. Details may be found in [128].

Take the L^2 inner-product of (3.1.1) and U, use Green's formula and take the real part to get

$$\frac{1}{2}\frac{d}{dt}|U|_{L^2}^2 = R|U|_{L^2}^2 - \|U\|_1^2 - |U|_{L^4}^4. \tag{3.4.24}$$

Now consider the three possible cases for R.

For $R < 0$ we have trivial asymptotic dynamics. From equation (3.4.24) we find $\forall\, t > 0$

$$|U(t)|_{L^2}^2 \leq |U^0|_{L^2}^2 e^{Rt}$$

and so for all $U^0 \in L^2$ we have exponential decay to the zero solution. For $R = 0$ we have trivial asymptotic dynamics. From equation (3.4.24) we find $\forall\, t > 0$

$$|U(t)|_{L^2}^2 \leq \frac{|U^0|_{L^2}^2|\Omega|}{|\Omega| + 2|U^0|_{L^2}^2},$$

and so for all $U^0 \in L^2$ we have algebraic decay to the zero solution. For $R > 0$ we proceed by noting that for all $s \in \mathbb{R}$ the following inequality holds:

$$\frac{1}{2}s^4 - 2Rs^2 \geq -2R^2, \tag{3.4.25}$$

from which

$$\frac{1}{2}|U|_{L^4}^4 - 2R|U|_{L^2}^2 \geq -2R^2|\Omega|.$$

Applying this to the L^4 norm in (3.4.24) to bound the L^2 norm we find

$$\frac{d}{dt}|U|_{L^2}^2 + 2\|U\|_1^2 + |U|_{L^4}^4 + 2R|U|_{L^2}^2 \leq 2R^2|\Omega|. \tag{3.4.26}$$

Thus by the standard Gronwall lemma 3.4.5

$$|U(t)|_{L^2}^2 \leq |U(0)|_{L^2}^2 e^{-Rt} + R|\Omega|\left(1 - e^{-Rt}\right) \tag{3.4.27}$$

for all $t \geq 0$. Whence,

$$\limsup_{t\to\infty} |U(t)|_{L^2}^2 \leq \rho'^2 \quad \text{where} \quad \rho'^2 := R|\Omega|. \tag{3.4.28}$$

Therefore the ball, $\mathcal{B}_0(\rho_0)$, $\rho_0 > \rho'$, is positively invariant and absorbing for the semigroup $S(t)$. For $B \subset \mathcal{B}_0(\rho)$ then $\forall\, t > t_0 = t_0(\rho, \rho_0)$

$$S(t)B \subset \mathcal{B}_0(\rho_0), \quad \rho_0 > \rho', \tag{3.4.29}$$

where

$$t_0 = \begin{cases} 0 & \text{if } \rho < \rho_0, \\ \frac{1}{R}\log\frac{\rho^2}{\rho_0^2-(\rho')^2} & \text{if } \rho > \rho_0 \end{cases}$$

From which the Theorem is proved.□

- For the remainder of the chapter we shall consider $R > 0$ for the continuous Ginzburg–Landau equation (3.1.1) since for $R \leq 0$ we have trivial dynamics.
- We can integrate (3.4.26) between $t > t_1$ and $t + r$ for arbitrary $r > 0$ to get

$$\int_t^{t+r} \{2R\|U\|_1^2 + |U|_{L^4}^4 + 2R|U|_{L^2}^2\}\, ds \leq \rho_0^2 + 2rR^2|\Omega| \tag{3.4.30}$$

for all $t \geq t_1$ and for all $r > 0$.

Absorbing Sets in H^1

Theorem 3.4.4 (C3) *There exists a constant $\rho_1 = \rho_1(R) > 0$ such that $\mathcal{B}_1(\rho_1)$ is absorbing and positively invariant for the semi-group $\{S(t)\}_{t\geq 0}$.*

Proof Another energy equation is sought, details again may be found in [128]. Take the Dirichlet inner-product of (3.1.1) with U, use Green's theorem and take the real part to get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|U\|_1^2 + |\Delta U|_{L^2}^2 - R\|U\|_1^2 &= \text{Re}\int_\Omega |U|^2 U \Delta \overline{U} dx \\ &\leq 3(1+\mu^2)^{1/2}\int_\Omega |U|^2|\nabla U|^2 dx. \end{aligned} \tag{3.4.31}$$

Now use Schwarz's inequality, the Sobolev embedding theorem, interpolation inequalities and complete the square to bound the right–hand side of (3.4.31) to get

$$\frac{d}{dt}\|U\|_1^2 \leq (2R + 9C(1+\mu^2)|U|_{L^4}^4)\,\|U\|_1^2 + |U|_{L^2}^2 - |\Delta U|_{L^2}^2 \tag{3.4.32}$$

where C is a positive constant.

Now apply the uniform Gronwall inequality 3.4.6 to (3.4.32) with $y = \|U\|_1^2$, $g = (2R + 9C(1+\mu^2)|U|_{L^4}^4)$, $h = |U|_{L^2}^2$ and using the bounds found in (3.4.30) to get

$$a_1 = 2Rr + 9C(1+\mu^2)\left(\rho_0^2 + 2rR^2|\Omega|\right) \tag{3.4.33}$$

$$a_2 = \rho_0^2 \tag{3.4.34}$$

$$a_3 = \frac{1}{2R}\left(\rho_0^2 + 2rR^2|\Omega|\right). \tag{3.4.35}$$

Thus:

$$\|U(t)\|_1^2 \leq \left(\frac{a_3}{r} + a_2\right) \exp(a_1) \tag{3.4.36}$$

for $t \geq t_1 + r$ where $r > 0$ is arbitrary and $U^0 \in \mathcal{B}$. Therefore we find that the ball $\mathcal{B}_1(\rho_1)$ with

$$(\rho_1)^2 = \rho_0^2 + \left(\frac{a_3}{r} + a_2\right) \exp(a_1) \tag{3.4.37}$$

is absorbing in H^1 for the semigroup $S(t)$.□

We now conclude the existence of a global attractor.

Theorem 3.4.5 (C4) *The dynamical system given by the complex Ginzburg–Landau equation* (3.1.1) *possesses a global attractor* $\mathcal{A}$,

$$\mathcal{A} = \omega\left(\mathcal{B}_1(\rho_1)\right).$$

Proof Theorems 3.4.3 and 3.4.4 provide all we require to satisfy all the conditions for Theorem 1.2.1 and we conclude that the complex Ginzburg–Landau equation possesses a global attractor with basin of attraction the whole of L_2. □

One Dimensional Alternatives

We now present two alternatives to the H^1 analysis given in Theorem 3.4.4 applicable in $p = 1$ dimension.

• First it was noted by Süli [126] that Temam uses the general Sobolev embedding theorem, and that the 1 dimensional version could be used instead. Note the following inequalities for $u \in H^1$ in one dimension using C to denote generic constant $C > 0$.

$$\begin{aligned}
|u|_{L^4} &\leq |u|_{L^\infty}^{1/2}|u|_{L^2}^{1/2} && (3.4.38)\\
|u|_{L^\infty} &\leq C|u|_{L^2}^{1/2}|u_x|_{L^2}^{1/2} && (3.4.39)\\
|u_x|_{L^2} &\leq C|u|_{L^2}^{1/2}\|u\|_{H^2}^{1/2} && (3.4.40)\\
|u|_{L^4} &\leq C|u_x|_{L^\infty}^{3/4}\|u\|_{H^2}^{1/4} \leq |u|^{3/8}\|u\|_{H^2}^{5/8} && (3.4.41)
\end{aligned}$$

where we have used (3.4.40) to get the last inequality in (3.4.41). Note that by (3.4.38, 3.4.39) and (3.4.40) we have

$$|u|_{L^4} \leq C|u|_{L^2}^{3/4}|u_x|_{L^2}^{1/4} \leq C|u|_{L^2}^{3/4}\left(|u|_{L^2}^{1/8}\|u\|_{H^2}^{1/8}\right) = C|u|_{L^2}^{7/8}\|u\|_{H^2}^{1/8}. \tag{3.4.42}$$

Combining (3.4.41) and (3.4.42) we see that

$$|u|_{L^4}^2|u_x|_{L^4}^2 \leq C\left(|u|_{L^2}^{7/4}\|u\|_{H^2}^{1/4}\right)\left(|u|_{L^2}^{3/4}\|u\|_{H^2}^{5/4}\right) = C|u|_{L^2}^{5/2}\|u\|_{H^2}^{3/2}. \tag{3.4.43}$$

In which case (3.4.32) would become, after an application of Hölder's inequality,

$$\frac{d}{dt}\|U\|_1^2 \leq 2R\|U\|_1^2 + \frac{3}{2}C\left((1+\mu^2)|U|_{L^2}^8 + 1\right)|U|_{L^2}^2 - \frac{1}{4}|\Delta U|_{L^2}^2$$

to which we could apply the uniform Gronwall Lemma 3.4.6 as in Theorem 3.4.4.

• The other alternative may be found in Doering et al [41]. They treat the non–linear term in a more delicate fashion and get bounds which depend on the parameters. In particular if we use the L^2 analysis as outlined above, so that ρ_0 is as defined above in (3.4.29) then we get the following bounds on the H^1 semi–norm :

- For arbitrary ν and $|\mu| \leq \sqrt{3}$ they find

$$\|U\|_1^2 \leq R\rho_1^2.$$

- For arbitrary ν and $|\mu| > \sqrt{3}$ they find

$$\|U\|_1^2 \leq \left(\frac{a_3}{r} + a_2\right)e^{a_1},$$

where

$$a_1 = R + \delta\rho_1^2 + 2\delta^2\rho_1^4,$$

$$a_2 = (R + \delta\rho_1^2 + 2\delta^2\rho_1^4)2\rho_1^2 r,$$

$$a_3 = (\rho_1^2 + 2rR^2|\Omega|)/2rR;$$

$$\delta = \max\left\{0, -2 + (1+\mu^2)^{1/2}\right\}$$

and r is an arbitrary positive constant.

- For arbitrary μ and $\nu = 0$ they find the L_∞ bound

$$|U|_\infty \leq R.$$

The case $\nu = 0$ is a special case for the Ginzburg–Landau equation as it corresponds to zero complex diffusion.

3.4.2 Gevrey Class

In **C1** (Theorem 3.4.2) we proved the existence and uniqueness of a solution $U(t)$ to (3.4.22) in L^2 and hence the existence of a bounded semi–group $S(t) : L^2 \to L^2$.

We would now like to know whether solutions to the Ginzburg–Landau equation become more regular in time through the smoothing action of the linear semigroup $L(t)$ (see Theorem 3.2.9).

Recall from (3.2.8) that solutions to the Ginzburg–Landau equation (3.4.22) may be written as

$$U(t) = L(t)U^0 + \int_0^t L(t-s)F_0(S(s)U^0)\, ds.$$

For $U^0 \in H^\alpha$ we take the H^β norm and apply Theorem 3.2.9 to find for $\beta \geq \alpha$ that

$$\|U(t)\|_{H^\beta} \leq Ct^{-(\beta-\alpha)/2}\|U^0\|_{H^\alpha} + C\int_0^t (t-s)^{-(\beta-\alpha)/2}\|F_0(S(s)U^0)\|_{H^\alpha}\, ds.$$

Thus, for example, in $p = 1$ dimension let us suppose we are given initial data $U^0 \in H^1$. We find that $U(t) \in H^2$ for $t > 0$:

$$\begin{aligned}
\|U(t)\|_{H^2} &\leq Ct^{-1/2}\|U^0\|_{H^1} + C\int_0^t (t-s)^{-1/2}\|F_0(S(s)U^0)\|_{H^1}\, ds \\
&\leq Ct^{-1/2}\|U^0\|_{H^1} + C\int_0^t (t-s)^{-1/2}\|F_0(S(s)U^0) - 0\|_{H^1}\, ds.
\end{aligned}$$

We now use that $F_0(S(s)0) = 0\ \forall s > 0$ and the H^1 Lipschitz property of F_0 of Lemma 3.5.2

$$\begin{aligned}
\|U(t)\|_{H^2} &\leq Ct^{-1/2}\|U^0\|_{H^1} + C\int_0^t (t-s)^{-1/2}\|F_0(S(s)U^0) - F_0(S(s)0)\|_{H^1}\, ds \\
&\leq Ct^{-1/2}\|U^0\|_{H^1} + C_1\int_0^t (t-s)^{-1/2}\|S(s)U^0\|_{H^1}\, ds.
\end{aligned}$$

Now by Gronwall's lemma 3.4.4 we have that

$$\|U(t)\|_{H^2} \leq C_2 t^{-1/2}.$$

Returning to both $p = 1$ and $p = 2$ dimensions if we could prove a sequence of Lipschitz estimates for F_0 in H^α, $\alpha = 1, 2, 3, \cdots, \beta - 1$, then we could prove that there exists T^* such that $U(t) \in H^\beta$ for all $t > T^*$.

Such estimates are called "ladder estimates" as each step depends on the next and Bartuccelli et al [9, 8] have had success with this approach. An overview of the "ladder estimates" for the Ginzburg–Landau equation may be found in [102].

An alternative to proving such "ladder estimates" is to prove a stronger result: that the solutions to the Ginzburg–Landau equation are of Gevrey class for some regularity τ (see (3.3.21)). This has been proved by Doelman and Titi [40] for the Ginzburg–Landau equation (3.1.1) with cubic non–linearity and by Duan et al [42] for the generalized Ginzburg–Landau equation.

Theorem 3.4.6 *If $U^0 \in H^1$ then there exists $T_* = T_*(\|U^0\|_{H^1})$ such that*

$$U(t) \in G_t = D(\widetilde{A}_0^{\frac{1}{2}} e^{t\widetilde{A}_0^{\frac{1}{2}}}), \quad t \in (0, T_*).$$

Furthermore $\exists$ σ independent of t and U^0 such that

$$U(t) \in G_\sigma, \quad \forall t \geq T$$

Proof See either of Doelman and Titi [40] or Duan et al [42].□

3.5 Low Dimensional Dynamics

Although it was shown in Section 3.4.1 that there exists a global attractor for the Ginzburg–Landau equation, the method of proof gave no estimate on the dimension of the attractor. It is conceivable that $\mathcal{A}$ is an infinite dimensional attractor, in which case we could justifiably question any attempt to approximate the dynamics of the attractor by a finite–dimensional system. For the complex Ginzburg–Landau equation, and many other equations such as the Kuramoto–Sivashinsky equation, the existence of an inertial manifold has been proved. We recall that an inertial manifold is a finite–dimensional positively invariant exponentially attracting manifold. This has two important consequences: (i) since it is exponentially attracting any interesting transient behaviour is contained in the inertial manifold; and (ii) since the global attractor is contained in the inertial manifold it is finite dimensional.

Indeed Ghidaglia and Héron [55] and Doering et al [41] obtain upper bounds on the Lyapunov dimension of the attractor and hence via the Kaplan–Yorke conjecture on the Hausdorff dimension (see section 1.2.3 and definitions 1.2.30 and 1.2.31). Lower bounds on the Lyapunov dimension of the attractor are found by considering exact solutions. More recently Kukavica [87] has proved that solutions to the Ginzburg–Landau equation (3.1.1) are completely determined by the values at two points. He uses this remarkable result to bound the fractal set of stationary solutions.

Numerically the finite dimensionality of the Ginzburg–Landau equation (3.1.1) has been investigated by Keefe [84] and Rodriguez et al [110, 111] among others.

3.5.1 An Inertial Manifold

Note : The analysis of this section on inertial manifolds and the cone condition is valid in $p = 1$ dimensions only.

Recall that an inertial manifold is an exponentially attracting, positively invariant, finite dimensional manifold which contains the global attractor. The first proof of the existence of an inertial manifold for the Ginzburg–Landau equation was given by Doering et al [41].

We shall prove the existence of an inertial manifold for the Ginzburg–Landau equation (3.1.1) by using the framework laid out in Jones and Stuart [83], applied to partial differential equations [83, Section 4]. We use this analysis as it is based on mappings in Banach spaces and hence is readily applicable to the fully discrete cases – see Chapter 5.

Recall the abstract evolution equation (3.2.3) in the Hilbert space X and let A be a sectorial operator. We define P_m to be the projection onto the eigenfunctions of A associated with the first m eigenvalues and we define $Q_m = I - P_m$. Further we define the spaces Y and Z by

$$Y = P_m X \text{ and } Z = Q_m X,$$

so that

$$X = Y \oplus Z.$$

We seek an inertial manifold $\mathcal{M}$ which is the graph of a function $\Phi : Y \to Z$,

$$\mathcal{M} = graph(\Phi).$$

The function Φ relates the high wave numbers to the low wave numbers and is often termed the "slaving function". (The space of high wave numbers Z is slaved to the space of low wave numbers Y).

Doering et al [41] prove the existence of the function Φ by proving that a cone condition holds (see Definition 3.5.1 below). This is the most common approach, see for example [29, 47]. In Jones and Stuart [83] the existence of Φ is proved by a contraction mapping argument and the cone condition is not explicitly used.

There are two elements in proving the existence of an inertial manifold: (i) the sectorial operator A must have certain spectral properties and (ii) the non–linear function is assumed to satisfy certain global Lipschitz properties. Such global bounds are usually obtained by using a suitable cut–off function and then considering the "prepared equation": as for example in [22, 47] and [83].

Next we give the conditions we are required to show in order to obtain the existence of an inertial manifold.

Assumptions E

- The operator A is sectorial;
- $\exists \gamma \geq 0, \beta \in [0,1)$ and $E(\sigma) > 0$ such that the nonlinear function $F : X^\gamma \to X^{\gamma-\beta}$ satisfies

$$\begin{aligned} \|F(u)\|_{X^{\gamma-\beta}} &\leq E(\sigma) \quad \forall u \in \mathcal{B}_\gamma(\sigma) \\ \|F(u) - F(v)\|_{X^{\gamma-\beta}} &\leq E(\sigma)\|u - v\|_{X^\gamma} \quad \forall u, v \in \mathcal{B}_\gamma(\sigma); \end{aligned}$$

- the eigenvalues of A, $\{\lambda_i\}$, satisfy
  a) $0 < \lambda_1 \leq \lambda_2 \cdots$,
  b) the *spectral gap* condition, i.e. for any $K_3, K_4 > 0$ there exists an integer q such that
  $$\lambda_{q+1}^{1-\beta} \geq K_3, \qquad \lambda_{q+1} - \lambda_q \geq K_4 \lambda_{q+1}^\beta;$$
  where $\beta \in [0,1)$,
- the equation generates a Lipschitz continuous semi–group $S_0(t) : X^\gamma \mapsto X^\gamma$ and there exists a $\rho > 0$ such that the ball $\mathcal{B}_\gamma \subset X^\gamma$ is absorbing.

We now state the existence theorem for an inertial manifold.

Theorem 3.5.1 *Under assumptions E the abstract evolution equation* (3.2.3) *has an inertial manifold: an exponentially attracting, positively invariant finite-dimensional manifold which can be represented as a graph of* $\Phi : Y \to Z$,

$$\mathcal{M} = \mathrm{graph}\,(\Phi)$$

within the ball $\mathcal{B}_\gamma(\rho) \subset X^\gamma$.

Proof For a proof of this Theorem see [48] or [83, Sections 2 and 4].□

We now proceed to use this Theorem to prove the existence of an inertial manifold for the Ginzburg–Landau equation in one space dimension. Recall that the eigenvalues of the linear operator $\widetilde{A}_0$ are given by $\{\widetilde{\Lambda}_k\}$.

First let us define the standard projections and decomposition of the space $X = L^2$. Let P_m denote the projection onto the eigenfunctions Ψ_k of $\widetilde{A}_0$ such that $|k| \leq m$ and let $Q_m = I - P_m$. Thus

$$P_m : L^2 \to \mathrm{Span}\,\{\Psi_{-m}, \cdots, \Psi_0, \cdots, \Psi_m\},$$

and may be thought of as the projection onto low wave numbers. Similarly Q_m may be thought of as the projection onto high wave numbers. Let

$$Y = P_m X, \quad \text{and} \quad Z = Q_m X,$$

so that

$$X = Y \oplus Z.$$

Lemma 3.5.1 (Spectral Gap) *For any* $K_1, K_2 > 0$ *there exists* $q \in \mathbb{N}$ *such that the eigenvalues* $\widetilde{\Lambda}_{q+1}$ *and* $\widetilde{\Lambda}_q$ *satisfy*

$$\widetilde{\Lambda}_{q+1} \geq K_1, \quad \textit{and} \quad \widetilde{\Lambda}_{q+1} - \widetilde{\Lambda}_q \geq K_2.$$

Proof The eigenvalues $\widetilde{\Lambda}_k$ are given by $\widetilde{\Lambda}_k = 1 + 4k^2\pi^2$, so we assume without loss of generality that $K_1 > 1$.

Pick q so that

$$q \geq \max\left\{\frac{\sqrt{K_1 - 1}}{2\pi} - 1, \frac{K_2}{8\pi^2} - \frac{1}{2}\right\}. \tag{3.5.44}$$

Then,

$$\widetilde{\Lambda}_{q+1} = 1 + 4(q+1)^2\pi^2 \geq 1 + 4\pi^2\frac{K_1 - 1}{4\pi^2} = K_1,$$

and

$$\widetilde{\Lambda}_{q+1} - \widetilde{\Lambda}_q = 4(q+1)^2\pi^2 - 4q^2\pi^2 = 8q\pi^2 + 4\pi^2 \geq K_2. \quad \square$$

We now turn our attention to the function F_0. Our proof of the following lemma holds only in one dimension.

Lemma 3.5.2 *Let $\rho > 0$ be given. Then there exists $C = C(\rho)$ such that the non-linear function F_0 defined by (3.4.23) satisfies in one spatial dimension*

$$||F_0(u)||_{H^1} \leq C \ \forall u \in \mathcal{B}_1(\rho)$$

and

$$||F_0(u) - F_0(v)||_{H^1} \leq C||u - v||_{H^1} \ \forall u, v \in \mathcal{B}_1(\rho).$$

Proof Note that

$$||F_0(u) - F_0(v)||_{H^1}^2 = |F_0(u) - F_0(v)|_{L^2}^2 + ||F_0(u) - F_0(v)||_1^2$$

and by Lemma 3.4.3

$$|F_0(u) - F_0(v)|_{L^2}^2 \leq C(\rho)|u - v|_{L^2}^2.$$

Thus all that remains to prove is that the H^1 semi-norm of F_0 satisfies the Lipschitz property. For $u \in \mathcal{B}_1(\rho)$ we have that

$$\begin{aligned} & ||F_0(u) - F_0(v)||_1^2 && (3.5.45)\\ = & \int_\Omega \left|\nabla\left(\widetilde{R}u - (1+i\mu)|u|^2u - (\widetilde{R}v - (1+i\mu)|v|^2v)\right)\right|^2 \, dx \\ \leq & \ 2|\widetilde{R}|^2 \int_\Omega |\nabla(u-v)|^2 dx \ + 2(1+\mu^2)\int_\Omega \left|\nabla\left(|u|^2u - |v|^2v\right)\right|^2 \, dx && (3.5.46)\end{aligned}$$

where we have used that

$$(a+b)^2 \leq 2a^2 + 2b^2, \quad a, b \in \mathbb{R}. \qquad (3.5.47)$$

Note that

$$\begin{aligned} & \left|(2|u|^2 - 2|v|^2)\nabla u + (u^2 - v^2)\nabla\overline{u}\right|^2 && (3.5.48)\\ = & \ \left|\left[2(u-v)\overline{u} + 2v(\overline{u} - \overline{v})\right]\nabla u + (u-v)(u+v)\nabla\overline{u}\right|^2 \\ \leq & \ \left|2|\overline{u} + v||u - v||\nabla u| + |u - v||u + v||\nabla\overline{u}|\right|^2 \\ \leq & \ \left|2|\overline{u} + v| + |u + v|\right|^2 |\nabla u|^2 |u - v|^2 && (3.5.49)\end{aligned}$$

and consider the last integral on its own and use (3.5.47) to find

$$
\begin{aligned}
&\int_\Omega \left|\nabla\left(|u|^2u-|v|^2v\right)\right|^2 dx \\
&= \int_\Omega \left|(2|u|^2\nabla u+u^2\nabla\overline{u})-(2|v|^2\nabla v+v^2\nabla\overline{v})\right|^2 dx \\
&\leq \int_\Omega \left|(2(|u|^2-|v|^2)\nabla u+(u^2-v^2)\nabla\overline{u}+2|v|^2\nabla u+v^2\nabla\overline{u}-2|v|^2\nabla v-v^2\nabla\overline{v})\right|^2 dx \\
&\leq 2\int_\Omega \left|2(|u|^2-|v|^2)\nabla u+(u^2-v^2)\nabla\overline{u}\right|^2 dx \\
&\quad +2\int_\Omega \left|2|v|^2(\nabla u-\nabla v)+v^2(\nabla\overline{u}-\nabla\overline{v})\right|^2 dx \\
&\leq 2\int_\Omega \left|2|\overline{u}+v|+|u+v|\right|^2|\nabla u|^2|u-v|^2dx+2\int_\Omega \left|2|v|^2+v^2\right|^2|\nabla u-\nabla v|^2dx.
\end{aligned}
$$

When in $p=1$ space dimension, for $u\in\mathcal{B}_1(\rho)$ we may bound the L^∞ norm to get that

$$\int_\Omega \left|\nabla\left(|u|^2u-|v|^2v\right)\right| dx \leq C_1(\rho)|u-v|_1^2+C_2(\rho)\|u-v\|_1. \tag{3.5.50}$$

The lemma is finally proved by combining (3.5.46) and (3.5.50). □

Theorem 3.5.2 *There exists an inertial manifold $\mathcal{M}$ for the one-dimensional Ginzburg-Landau equation* (3.4.22) *which may be represented as the graph of $\Phi: Y\to Z$ within the absorbing ball $\mathcal{B}_1(\rho_1)$.*

Proof Lemmas 3.5.2 and 3.5.1 and Theorems 3.3.2 and 3.4.4 ensure that all the assumptions of Theorem 3.5.1 are satisfied. □

As stated above another method of proof is to first prove that a "cone condition" holds.

3.5.2 The Cone Condition

We now define the cone condition for the abstract evolution equation (3.2.3) in the Hilbert space X. Let $\mathcal{C}_{m,\gamma}$ be the cone

$$\mathcal{C}_{m,\gamma} := \left\{v\in L^2 : |Q_mv|_{L^2}\leq\gamma|P_mv|_{L^2}\right\},$$

and let $u_1(t)$, $u_2(t)$ be two solutions to the equation (3.2.3) at time t.

Definition 3.5.1 Given solutions $u_1(t)$, $u_2(t)$ and cone $\mathcal{C}_{m,\gamma}$ for an evolution equation of the form (3.2.3) then the *cone condition* is said to be satisfied if

a) $u_1(0) \in u_2(0) + \mathcal{C}_\gamma$ then $u_1(t) \in u_2(t) + \mathcal{C}_\gamma \quad \forall t \geq 0$;

b) if $u_1(0) \notin u_2(0) + \mathcal{C}_\gamma$ then

- either : $u_1(t_0) \in u_2(t_0) + \mathcal{C}_\gamma$ for some $t_0 > 0$ and then $u_1(t) \in u_2(t) + \mathcal{C}_\gamma \quad \forall t \geq t_0$
- or : $u_1(t) \notin u_2(t) + \mathcal{C}_\gamma \quad \forall t \geq 0$ in which case $u_1(t) - u_2(t)$ decays exponentially to 0 as $t \to \infty$.

In other words the cone condition states that

- if u_1^0 is inside the cone about u_2^0 then it remains inside the cone for all future time
- and if u_1^0 is not inside the cone about u_2 then either it enters the cone after some finite time and remains inside, or u_1 is attracted exponentially to u_2.

We now prove the cone condition for the solutions on the global attractor of the Ginzburg–Landau equation.

Theorem 3.5.3 *Let* $U^0, V^0 \in \mathcal{B}_1(\rho_1)$ *and let*

$$B := \max_{t \geq 0} \{|U(t)|^2_{L^\infty}, |V(t)|^2_{L^\infty}\}$$

and choose M *from Lemma 3.5.1 so that*

$$\Lambda_M^{1/2} > \max\left\{R^{1/2}, 3B(1+\mu^2)^{1/2}/\Lambda_1^{1/2}\right\}.$$

Then U, V *satisfy the cone condition with cone*

$$C_{M,1} = \{v \in L^2 : |Q_M v|_{L^2} \leq |P_M v|_{L^2}\}.$$

Proof The proof here uses the outline by Doering et al [41].

Consider the difference of two solutions $U(t), V(t)$ to the Ginzburg–Landau equation (3.1.1). Then $U(t) - V(t)$ satisfies

$$\frac{d}{dt}(U(t) - V(t)) = RA(t) + (1+i\nu)(U(t) - V(t))_{xx} - (1+i\mu)\{|U|^2U - |V|^2V\}.$$

Consider the projection $p(t) := P_M(U(t) - V(t))$ onto low wave numbers and the projection $q(t) := Q_M(U(t) - V(t))$ onto high wave numbers which satisfy:

$$\frac{d}{dt}p = Rp + (1+i\nu)p_{xx} - (1+i\mu)P_M\left\{|U|^2U - |V|^2V\right\} \tag{3.5.51}$$

and

$$\frac{d}{dt}q = Rq + (1+i\nu)q_{xx} - (1+i\mu)Q_M\left\{|U|^2U - |V|^2V\right\}. \tag{3.5.52}$$

Take the L^2 inner-product of (3.5.51) with p and (3.5.52) with q, use integration by parts noting that

$$\int_0^1 \overline{p_x}(p_x) \le \Lambda_M|p|_{L^2}^2$$

and

$$\int_0^1 \overline{q_x}(q_x) \ge \Lambda_{M+1}|q|_{L^2}^2$$

and take the real part to get

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}|p|_{L^2}^2 &\ge R|p|_{L^2}^2 - \Lambda_M|p|_{L^2}^2 - \mathrm{Re}\left\{(1+i\mu)\int_0^1 \overline{p}P_M\left(|U|^2U - |V|^2V\right)\right\} && (3.5.53)\\
\frac{1}{2}\frac{d}{dt}|q|_{L^2}^2 &\le R|q|_{L^2}^2 - \Lambda_M|q|_{L^2}^2 - \Lambda_1^{1/2}(2\Lambda_M^{1/2} + \Lambda_1^{1/2})|q|_{L^2}^2\\
&\quad -\mathrm{Re}\left\{(1+i\mu)\int_0^1 \overline{q}Q_M(|U|^2U - |V|^2V)\right\}, && (3.5.54)
\end{aligned}$$

where in (3.5.54) we also use that

$$\Lambda_{M+1} = 4\pi^2(M+1)^2 = \Lambda_M + 2\Lambda_M^{1/2}\Lambda_1^{1/2} + \Lambda_1.$$

Now define $\Theta_0(t) := |q(t)|_{L^2}^2 - |p(t)|_{L^2}^2$ which we use to determine if the cone condition is satisfied. From equations (3.5.53) and (3.5.54) we have that $\Theta_0(t)$ satisfies

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\Theta_0(t) &\le (R - \Lambda_M)\Theta_0 - \Lambda_1^{1/2}\left(2\Lambda_M^{1/2} + \Lambda_1^{1/2}\right)|q|_{L^2}^2\\
&\quad - \mathrm{Re}\left\{(1+i\mu)\int_0^1 \overline{q}Q_M(|U|^2U - |V|^2V) - \overline{p}P_M(|U|^2U - |V|^2V)\right\}.
\end{aligned} \tag{3.5.55}$$

Now consider the non–linear term separately.

$$\begin{aligned}
&- \mathrm{Re}\left\{(1+i\mu)\int_0^1 \overline{q}Q_M(|U|^2U - |V|^2V) - \overline{p}P_M(|U|^2U - |V|^2V)\right\}dx\\
&\le (1+\mu^2)^{1/2}\int_0^1 |\overline{q}Q_M(|U|^2U - |V|^2V) - \overline{p}P_M(|U|^2U - |V|^2V)|\,dx\\
&\le (1+\mu^2)^{1/2}\int_0^1 |\overline{q}Q_M(|U|^2U - |V|^2V)|\,dx\\
&\quad + (1+\mu^2)^{1/2}\int_0^1 |\overline{p}P_M(|U|^2U - |V|^2V)|\,dx.
\end{aligned}$$

By the same analysis as in Lemma 3.4.1 we get that

$$\begin{aligned}
&(1+\mu^2)^{1/2}\int_0^1 |\overline{q}Q_M(|U|^2U-|V|^2V)|\,dx\\
\le\;& 2(1+\mu^2)^{1/2}\int_0^1 |\overline{q}q\left(|U|^2+|V|^2\right)+\overline{q}^2UV|\,dx.\\
\le\;& 3(1+\mu^2)^{1/2}\left(|U|^2_{L^\infty}+|V|^2_{L^\infty}\right)|q|^2_{L^2}
\end{aligned}$$

and similarly for other integral. Hence returning to (3.5.55) and using the definition of B in the statement of the theorem we have

$$\frac{1}{2}\frac{d}{dt}\Theta_0(t)\le(R-\Lambda_M)\Theta_0-\Lambda_1^{1/2}\left(2\Lambda_M^{1/2}+\Lambda_1^{1/2}\right)|q|^2_{L^2}+3B(1+\mu^2)^{1/2}\left(|q|^2_{L^2}+|p|^2_{L^2}\right). \tag{3.5.56}$$

By the choice of M we see that for $|q|^2_{L^2}\ge|p|^2_{L^2}$ we can use the second term to control the rest, so that

$$\frac{1}{2}\frac{d}{dt}\Theta_0<(R-\Lambda_M)\Theta_0-\Lambda_1|q|^2_{L^2}$$

and so outside the cone we have exponential decay and in addition on the surface of the cone that $\frac{d}{dt}\Theta_0<0$ hence once the cone condition is satisfied it remains satisfied for all time. □

Notes

- The cone condition is proved for any two solutions of the Ginzburg–Landau equation by [128, Chapter VIII] in the norm of $D(A^\alpha)$ with $\frac{1}{8}\le\alpha\le\frac{3}{8}$.

- Theorem 3.5.3 yields a bound on the dimension of the attractor $\mathcal{A}$ in the sense that it bounds the number of Fourier modes required to span the attractor. Denoting the *Fourier dimension* by D_F we have that

$$D_F\le 2M+1 \tag{3.5.57}$$

where M is given in Theorem (3.5.3). We may estimate B using the 1 dimensional Sobolev embedding theorem and either the H^1 bound found as a result of Theorem 3.4.4 or the H^1 and L^∞ bound found by [41] also given after Theorem 3.4.4.

3.5.3 Lyapunov Dimension

As mentioned in the introduction to Section 3.5 Ghidaglia and Héron [55] and Doering et al [41] estimated the Lyapunov dimension (see (1.2.23)) to bound the Hausdorff dimension of the global attractor for the Ginzburg–Landau equation (3.1.1). Their method of proof is based on the analysis by Constantin and Foias [27]. We outline below elements required in Constantin et al's analysis, starting with some further notation and a couple of technical lemmas, which may also be found, for example, in [27].

Notation and Definitions:

- Let $\{V_i\}_{i=1}^{M}$ be a sequence of vectors in a Hilbert space X with inner product $< \bullet, \bullet >$ and induced norm $\| \bullet \|_X^2 =< \bullet, \bullet >$. Define $P(V_1, \cdots, V_M)$ to be the orthogonal projection

$$P(V_1, \cdots, V_M) : X \to \mathrm{Sp}\{V_1, \cdots, V_M\}.$$

- For $\{V_i\}_{i=1}^{M}$ in X we define the *tensor product* of $V_1, \cdots, V_m$ by

$$(V_1 \otimes \cdots \otimes V_M)(\xi_1, \ldots, \xi_M) = \prod_{i=1}^{M} < V_i, \xi_i >, \quad \forall \xi_1, \ldots, \xi_M \in X.$$

  The *exterior* or *wedge product* is then defined by

$$V_1 \wedge \cdots \wedge V_M := \sum_{\sigma} (-1)^{\sigma} V_{\sigma(1)} \otimes \ldots \otimes V_{\sigma(M)},$$

  where σ is a permutation of $\{1, \ldots, M\}$, $(-1)^{\sigma}$ is the sign of σ and the sum is extended to all such permutations. For further details see [16].

- Let T be a linear map with domain D_T, $T : D_T \to L^2$, then we define T_M to be the operator

$$T_M = \underbrace{(T \wedge I \wedge \cdots \wedge I)}_{M \text{ times}} + \underbrace{(I \wedge T \wedge \cdots \wedge I)}_{M \text{ times}} + \cdots + \underbrace{(I \wedge \cdots \wedge I \wedge T)}_{M \text{ times}}. \tag{3.5.58}$$

- We shall use a $\widehat{\ }$ to indicate that a term is omitted.

Lemma 3.5.3 *Given vectors $\{V_i\}_1^M$ and $\{W_i\}_1^M$ in the Hilbert space X, then*

$$< V_1 \wedge \cdots \wedge V_M, W_1 \wedge \cdots \wedge W_M > \; = \; det[< V_i, W_j >_{i,j=1,\ldots,M}]. \tag{3.5.59}$$

Proof See for example [16].□

Lemma 3.5.4 *For any W and $\{V_i\}_{i=1}^M \in L^2$, we have that*

$$\begin{aligned} &\|V_1 \wedge \cdots \wedge V_M\|_X^2 P(V_1, \cdots, V_M) W \\ = &\sum_{k=1}^{M} (-1)^{k-1} < W \wedge V_1 \wedge \cdots \wedge \widehat{V_k} \wedge \cdots \wedge V_M, V_1 \wedge \cdots \wedge V_M > V_k. \end{aligned} \tag{3.5.60}$$

Proof [27, Lemma 3.2]

Clearly the projection of W onto $\mathrm{Sp}(V_1, \cdots, V_M)$ is given by

$$P(V_1, V_2, \cdots, V_M) W = \sum_{k=1}^{M} \gamma_k(W) V_k$$

where the γ_k satisfy

$$< W, V_k >= \sum_{\ell=1}^{M} \gamma_\ell < V_\ell, V_k > . \tag{3.5.61}$$

Equation (3.5.60) follows by applying Cramer's rule to (3.5.61) to find γ_k and then using the relation (3.5.59). □

Lemma 3.5.5 *Let $T : X \to X$ be a linear map. Then provided that $V_1, \cdots, V_M$, are in the domain of T, we have that*

$$\langle T_M (V_1 \wedge \cdots \wedge V_M), V_1 \wedge \cdots \wedge V_M \rangle = \|V_1 \wedge \cdots \wedge V_M\|_X^2 \, Tr \, (T \cdot P(V_1, \cdots, V_M)), \tag{3.5.62}$$

where Tr denotes the trace of the operator as defined in [128].

Proof We outline the proof presented in [27, Lemma 3.3].

Let $\{W_\ell\}$ be an orthonormal basis for X such that

$$\mathrm{Sp}\ \{W_1, \cdots, W_\ell\} = \mathrm{Sp}\ \{V_1, \cdots, V_M\}.$$

From equation (3.5.60) we find for every ℓ

$$\begin{aligned} &\|V_1 \wedge V_2 \wedge \cdots \wedge V_M\|_X^2 W_\ell \\ = &\|V_1 \wedge \cdots \wedge V_M\|_X^2 P(V_1, \cdots, V_M) W_\ell \\ = &\sum_{k=1}^{M} (-1)^{k-1} < W_\ell \wedge V_1 \wedge \cdots \wedge \widehat{V_k} \wedge \cdots \wedge V_M, V_1 \wedge \cdots \wedge V_M > V_k. \end{aligned}$$

We apply the Laplace expansion of the determinant on the right–hand side to get

$$
\begin{aligned}
&\|V_1 \wedge V_2 \wedge \cdots \wedge V_M\|_X^2 W_\ell \\
&= \sum_{k=1}^{M}\sum_{m=1}^{M}(-1)^{k-1}(-1)^{m-1} < W_l, V_m > \\
&\quad \times < V_1 \wedge V_2 \wedge \cdots \wedge \widehat{V_k} \wedge \cdots \wedge V_M, V_1 \wedge \cdots \wedge \widehat{V_m} \wedge \cdots \wedge V_M > V_k .
\end{aligned}
\tag{3.5.63}
$$

Since the W_ℓ's span the space as V_m's we have

$$
\|V_1 \wedge \cdots \wedge V_M\|_X^2 \mathrm{Tr}\ (T \cdot P(V_1, \cdots, V_M)) = \|V_1 \wedge \cdots \wedge V_M\|_X^2 \sum_{l=1}^{M} < TW_\ell, W_\ell >,
$$

which if we apply T to (3.5.63) and take the inner product with W_ℓ we find

$$
\begin{aligned}
&\|V_1 \wedge \cdots \wedge V_M\|_X^2 \mathrm{Tr}\ (T \cdot P(V_1, \cdots, V_M)) \\
&= \sum_{k=1}^{M}\sum_{m=0}^{M}\sum_{l=0}^{M}(-1)^{k+m-2} < TV_k, W_l >< W_\ell, V_m > \\
&\quad \times < V_1 \wedge V_2 \wedge \cdots \wedge \widehat{V_j} \wedge \cdots \wedge V_M, V_1 \wedge \cdots \wedge \widehat{V_m} \wedge \cdots \wedge V_M > .
\end{aligned}
$$

Now use the fact that the W_ℓ were chosen to be an orthonormal basis to get:

$$
\begin{aligned}
&\|V_1 \wedge \cdots \wedge V_M\|_X^2 \mathrm{Tr}\ (T \cdot P(V_1, \cdots, V_M)) \\
&= \sum_{k=1}^{M}\sum_{m=1}^{M}(-1)^{k+m-2} < TV_k, V_m > \\
&\quad \times < V_1 \wedge V_2 \wedge \cdots \wedge \widehat{V_k} \wedge \cdots \wedge V_M, V_1 \wedge \cdots \wedge \widehat{V_m} \wedge \cdots \wedge V_M > .
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\|V_1 \wedge \cdots \wedge V_M\|_X^2 \mathrm{Tr}\ (T \cdot P(V_1, \cdots, V_M)) \\
&= \sum_{k=1}^{M}(-1)^{k-1} < TV_k \wedge V_1 \wedge \cdots \wedge V_M, V_1 \wedge \cdots \wedge V_M > \\
&= \quad < T_M(V_1 \wedge \cdots \wedge V_M), V_1 \wedge \cdots \wedge V_M > . \qquad \square
\end{aligned}
$$

Reconsider the evolution equation

$$
U_t = F(U) \tag{3.5.64}
$$

on the Hilbert space X, and let us assume we have proved the existence of a global attractor $\mathcal{A}$ for this equation. Recall from Definition 1.2.29 that the sum of the first m

global Lyapunov exponents $\mu_1 + \cdots + \mu_m$ for (3.5.64) is given by

$$\mu_1 + \cdots + \mu_m = \limsup_{t\to\infty} \frac{1}{t} \log \left\{ \sup_{U^0 \in \mathcal{A}} \sup_{\substack{\xi_i^0 \in X \\ \|\xi_i^0\| \le 1}} |L(t, U^0)\xi_1^0 \wedge \cdots \wedge L(t, U^0)\xi_m^0| \right\} \tag{3.5.65}$$

where $L(t, U^0)\xi_i^0$ is the solution at time t of the linear evolution equation

$$\frac{d}{dt}\xi_i = DF[U(t)]\xi_i, \tag{3.5.66}$$

with initial condition ξ_i^0.

The following theorem gives a more useful expression for the sum of the first m global Lyapunov exponents.

Theorem 3.5.4 *The sum of the first m global Lyapunov exponents for the evolution equation* (3.5.64) *is given by*

$$\sum_{i=1}^{m} \mu_i = \limsup_{t\to\infty} \frac{1}{t} \log \left\{ \sup_{U^0 \in \mathcal{A}} \sup_{|\xi_i|_2^2 \le 1} \exp \left[Re \int_0^t Tr(DF[U(s)] \cdot P(\xi_1(t), \cdots, \xi_m(t)))\, ds \right] \right\} \tag{3.5.67}$$

where $DF[U(t)]$ is the Fréchet derivative of F and $\xi_i(t) = L(t, U^0)\xi^0$ is the solution of the linear evolution equation (3.5.66).

Proof

Consider the time evolution of the m–form $(\xi_1(t) \wedge \cdots \wedge \xi_m(t))$

$$\begin{aligned} &\frac{d}{dt}(\xi_1(t) \wedge \cdots \wedge \xi_m(t)) \\ &= \frac{d}{dt}\xi_1 \wedge \cdots \wedge \xi_m + \cdots + \wedge \frac{d}{dt}\xi_m \\ &= DF[U(t)]\xi_1(t) \wedge \cdots \wedge \xi_m(t) + \cdots + \xi_1(t) \wedge \cdots \wedge DF[U(t)]\xi_m(t). \end{aligned} \tag{3.5.68}$$

Note that $P(\xi_1(t), \cdots, \xi_m(t))\, \xi_i(t) = \xi_i(t)$, so that

$$\begin{aligned} \frac{d}{dt}(\xi_1 \wedge \cdots \wedge \xi_m) &= DF[U(t)]P(\xi_1(t), \cdots, \xi_m(t))\, \xi_1 \wedge \cdots \wedge \xi_m + \cdots \\ &\quad + \xi_1 \wedge \cdots \wedge DF[U(t)]P(\xi_1(t), \cdots, \xi_m(t))\, \xi_m. \end{aligned}$$

Now consider an m–volume element:

$$\begin{aligned}
&\frac{d}{dt}\|\xi_1\wedge\cdots\wedge\xi_m\|_X^2 \\
&= \frac{d}{dt}\langle \xi_1\wedge\cdots\wedge\xi_m, \xi_1\wedge\cdots\wedge\xi_m\rangle \\
&= 2\mathrm{Re}\ \{< DF[U(t)]\cdot P(\xi_1(t),\cdots,\xi_m(t))\xi_1\wedge\cdots\wedge\xi_m, \xi_1\wedge\cdots\wedge\xi_m > +\cdots \\
&\qquad \cdots + < \xi_1\wedge\cdots\wedge DF[U(t)]\cdot P(\xi_1(t),\cdots,\xi_m(t))\xi_m, \xi_1\wedge\cdots\xi_m >\},
\end{aligned}$$

which by Lemma 3.5.5 becomes :

$$\frac{d}{dt}\|\xi_1\wedge\cdots\wedge\xi_m\|_X^2 = 2\|\xi_1\wedge\cdots\wedge\xi_m\|_X^2\mathrm{Re}\,\{\mathrm{Tr}(DF[U(t)]\cdot P(\xi_1(t),\cdots,\xi_m(t)))\}. \qquad (3.5.69)$$

If we solve (3.5.69) we find

$$\begin{aligned}
&\|\xi_1(t)\wedge\cdots\wedge\xi_m(t)\|_X \\
&= \|\xi_1^0\wedge\cdots\wedge\xi_m^0\|_X \exp\left[\mathrm{Re}\left(\int_0^t \mathrm{Tr}(DF[U(s)]\cdot P(\xi_1(t),\cdots,\xi_m(t)))ds\right)\right].
\end{aligned}$$

Substituting this into (3.5.65) we obtain (3.5.67). □

The lower and upper bounds on the sum of Lyapunov exponents are found by bounding (3.5.67) from below and above. For an upper bound we shall make use of the following lemma.

Lemma 3.5.6 *Let A be a linear positive unbounded self-adjoint operator in the Hilbert space X with compact inverse A^{-1}. Then, for any elements $\{\phi_j\}$ which are orthonormal in X we have*

$$\sum_{j=1}^m \langle A\phi_j, \phi_j\rangle \geq \lambda_1 + \cdots + \lambda_m,$$

where (λ_j) is the complete sequence of eigenvalues of A.

Proof See for example [128, p 302]. □

Lemma 3.5.7 *The linearised evolution of an arbitrary perturbation ξ to a solution U of the complex Ginzburg–Landau equation (3.1) is given by*

$$\frac{d}{dt}\xi = DF[U(t)]\xi,$$

where,

$$DF[U(t)]\xi := R\xi - (1+i\nu)A_0\xi - (1+i\mu)\left\{2|U(t)|^2\xi + U(t)^2\overline{\xi}\right\}. \qquad (3.5.70)$$

Proof Straightforward.□

Corollary 3.5.1 *The sum of the first m global Lyapunov exponents for the Ginzburg–Landau equation satisfies for m odd*

$$\mu_1 + \cdots + \mu_m \geq \sum_{k=-(m-1)/2}^{(m-1)/2} (R - \Lambda_k)$$

and for m even,

$$\mu_1 + \cdots + \mu_m \geq \sum_{k=-(m/2-1)}^{m/2} (R - \Lambda_k)$$

where Λ_k is the k^{th} eigenvalue of the linear operator A_0.

Proof See [55] or [41]. To achieve this lower bound we use the fact that $U \equiv 0$ is a stationary solution and thus is on the global attractor. Setting $U \equiv 0$ in (3.5.70) we get

$$\begin{aligned}
&\sup_{U^0} \sup_{|\xi_i|_2^2 \leq 1} \exp\left[\mathrm{Re} \int_0^t \mathrm{Tr}\,(DF[U(s)] \cdot P(\xi_1(t), \cdots, \xi_m(t))\, ds\right] \\
&\geq \sup_{|\xi_i|_2^2 \leq 1} \exp\left[\mathrm{Re} \int_0^t \mathrm{Tr}(DF[0] \cdot P(\xi_1, \cdots, \xi_m)) ds\right] \\
&\geq \exp\left[\mathrm{Re} \int_0^t \mathrm{Tr}(DF[0] \cdot P(\Psi_0, \Psi_1, \Psi_{-1}, \cdots) ds\right]
\end{aligned}$$

where $\{\Psi_i\}$ are the eigenfunctions of the linear operator A_0 defined in Theorem 3.3.1 and $P(\Psi_0, \Psi_1, \Psi_{-1}, \cdots)$ is the projection onto the first m such functions.

Noting that

$$DF[0]\xi = R\xi - (1 + i\nu) A_0 \xi,$$

we find that

$$\mathrm{Tr}\,(DF[0] \cdot P(\Psi_0, \Psi_1, \Psi_{-1} \cdots)) = \sum_{k=-(m-1)/2}^{(m-1)/2} < \Psi_k, [RI - (1 + i\nu) A_0] \Psi_k > \text{ if } m \text{ odd}$$

and

$$\mathrm{Tr}\,(DF[0] \cdot P(\Psi_0, \Psi_1, \Psi_{-1} \cdots)) = \sum_{k=-(m/2-1)}^{m/2} < \Psi_k, [RI - (1 + i\nu) A_0] \Psi_k > \text{ if } m \text{ even.}$$

Noting that

$$< \Psi_k, [RI - (1 + i\nu) A_0] \Psi_k >= R - (1 + i\nu) \Lambda_k,$$

and substituting into (3.5.67) we obtain the desired result.□

Corollary 3.5.2 *The sum of the first m–global Lyapunov exponents for the Ginzburg–Landau equation* (3.1) *satisfies :*

$$\mu_1 + \cdots + \mu_m \le \sum_{k=-(m-1)/2}^{(m-1)/2} (R + \delta |U|^2_{L^\infty} - \Lambda_k) \text{ if } m \text{ odd;}$$

$$\mu_1 + \cdots + \mu_m \le \sum_{k=-(m/2-1)}^{m/2} (R + \delta |U|^2_{L^\infty} - \Lambda_k) \text{ if } m \text{ even,}$$

where Λ_k is the kth eigenvalue of the linear operator A_0, and

$$\delta = \max\left\{0, -2 + (1+\mu^2)^{1/2}\right\}.$$

Proof We find the required upper bound by estimating the trace term in equation (3.5.67). Let $\{\Phi_k\}_1^m$ be a set of orthonormal vectors such that

$$\mathrm{Sp}\,(\Phi_1, \cdots, \Phi_m) = P_m\,(\xi_1(t), \cdots, \xi_m(t)),$$

where $\xi_1, \cdots, \xi_m$ are as in Theorem 3.5.4.
Then if m is odd

$$\mathrm{Re}\,\{\mathrm{Tr}\,(DF[U(t)] \cdot P\,(\xi_1(t), \cdots, \xi_m(t)))\} = \mathrm{Re} \sum_{k=-(m-1)/2}^{(m-1)/2} < \Phi_k, DF[U(t)]\Phi_k >, \tag{3.5.71}$$

and if m is even

$$\mathrm{Re}\,\{\mathrm{Tr}\,(DF[U(t)] \cdot P\,(\xi_1(t), \cdots, \xi_m(t)))\} = \mathrm{Re} \sum_{k=-(m/2-1)}^{m/2} < \Phi_k, DF[U(t)]\Phi_k > \tag{3.5.72}$$

where we have that

$$\begin{aligned} < \Phi_k, DF[U(t)]\Phi_k > \;&=\; \langle \Phi_k, (RI - A_0)\Phi_k\rangle - 2\,\langle \Phi_k, |U|^2\Phi_k\rangle \\ &\quad - \mathrm{Re}\left\{(1+i\mu)\left\langle \Phi_k, U^2\overline{\Phi}_k\right\rangle\right\}. \end{aligned} \tag{3.5.73}$$

We restrict attention to the last two terms in (3.5.73), and note that for any $\Phi \in L^2$

$$\begin{aligned} & -2 < \Phi, |U|^2\Phi > -\mathrm{Re}\left\{(1+i\mu) < \Phi, U^2\overline{\Phi} >\right\} & (3.5.74) \\ &= -2\int_0^1 \Phi|U|^2\overline{\Phi}dx - \mathrm{Re}\left\{(1+i\mu)\int_0^1 \overline{U}^2\Phi^2 dx\right\} \\ &\le -2\int_0^1 |U|^2|\Phi|^2 dx + |1+i\mu|\int_0^1 |U|^2|\Phi|^2 dx \\ &\le \delta|U|^2_{L^\infty}|\Phi|^2_{L^2} & (3.5.75) \end{aligned}$$

where $\delta = \max\{0, -2 + (1+\mu^2)^{1/2}\}$. Substituting (3.5.75) into (3.5.71) and (3.5.72) and using Lemma 3.5.6 we find the desired result. □

Notes

- When $\mu \leq \sqrt{3}$ we find $\delta = 0$, in which case the lower and upper bounds coincide and we have found the global Lyapunov exponents exactly.
- From the Kaplan–Yorke formula (see (1.2.23)) we immediately find bounds on the Lyapunov dimension, which in turn yields an upper bound on the Hausdorff dimension. For further details see [41] and [27].

3.5.4 Waves and Connections

The complex Ginzburg–Landau equation (3.1.1) possesses many non–trivial exact solutions. Possibly the simplest of these are the rotating wave solutions

$$U_m = a_m \exp[i(\Lambda_m^{1/2}x - \omega_m t)]; \tag{3.5.76}$$

where the amplitude a_m and period ω satisfy

$$|a_m|^2 = R - \Lambda_m, \tag{3.5.77}$$

$$\omega_m = R\mu + (\nu - \mu)\Lambda_m \tag{3.5.78}$$

and

$$\Lambda_m = 4m^2\pi^2, \ m = 0, \pm 1, \cdots$$

are the eigenvalues of A_0 given in Theorem 3.3.1.

Remarks

- For $m = 0$, ($\Lambda_m = 0$) we have the spatially homogeneous wave

$$U_0 = a_m e^{-i\omega_m t}.$$

- The m^{th} rotating wave comes into existence when the bifurcation parameter R satisfies $R = \Lambda_m$, and exists for all $R \geq \Lambda_m$ (see Lemma 3.5.8 below). This gives the bifurcation diagram in the $(R, |a_m|^2)$ plane shown in Figure 3.1 for $R \geq 0$. Each line corresponds to a rotating wave solution originating from the trivial solution $U \equiv 0$ (equivalently $a_m = 0$).

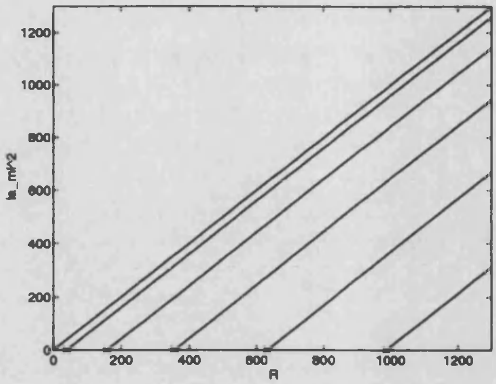

Figure 3.1: Bifurcation diagram for the rotating waves

- The bifurcation of the trivial solution which give rise to the mth rotating wave is a Hopf like bifurcation. We are unable to apply the standard Hopf bifurcation Theorem because we have multiple eigenvalues. Although we are able to prove necessary conditions for a Hopf bifurcation, see Lemma 3.5.8, and the rotating wave solutions give an explicit expression for the resulting periodic solutions it is possible that other solutions bifurcate from the same point. A more complete discussion may be found in Takáč [127].

Lemma 3.5.8 *The mth rotating wave U_m originates from the mth Hopf-like bifurcation of the zero solution.*

Proof Recall the Ginzburg-Landau equation (3.1) and linearise about a solution U to get the linearised evolution equation:

$$\epsilon_t = R\epsilon - (1 + i\nu)A_0\epsilon - (1 + i\mu)(2|U|^2\epsilon + U^2\bar{\epsilon}). \tag{3.5.79}$$

Thus linearising about the trivial solution $U \equiv 0$ solution we find

$$\epsilon_t = (RI - (1 + i\nu)A_0)\,\epsilon.$$

The linear operator $(RI - (1 + i\nu)A_0)$ has eigenvalues μ_m given by

$$\mu_m = R - (1 + i\nu)\Lambda_m.$$

To prove the result note that there is a pair of pure imaginary eigenvalues if and only if $R = \Lambda_m$ and that $\frac{d}{d\Lambda}(\text{ Re}\mu(\Lambda))|_{\Lambda=\Lambda_m} = -1.\Box$

Since the rotating waves are contained in the global attractor $\mathcal{A}_0$, Doering et al [41] are able to use the rotating wave solutions to obtain lower bounds on the dimension of the global attractor. These bounds are found by considering the stability of rotating wave U_m to linear perturbations in the ℓth mode ($m \neq \ell$). This leads to a neutral stability curve on which the wave U_m is neutrally stable to linear perturbations of wave number ℓ, $\ell \neq 0$. This curve is determined by

$$4\Lambda_m = \left(|a_m|^2 + \Lambda_m\right)^2 \left[(1+\nu^2)\Lambda_\ell + 2(1+\nu\mu)|a_m|^2\right] / \left[(|a_m|^2+\Lambda_m)^2 + \mu|a_m|^2 + \nu\Lambda_m\right]. \tag{3.5.80}$$

For perturbations in the 0th mode ($\ell = 0$) one may prove that the mth wave is not linearly unstable to any perturbation in that mode. From relation (3.5.80) it is possible to plot the neutral stability curves in the $(\sqrt{R}/2\pi, m)$ plane. The figures presented here have been produced with the aid of the numerical continuation program **Pitcon** [107]. The Figures 3.2 show the neutral stability curves for the same parameter values as chosen in [41]. The mth rotating wave U_m comes into existence at the point (m, m) and exists along the horizontal line. To find out if this mth wave is stable to linear perturbations in the ℓth mode one simply locates the ℓth curve along the diagonal counting from $(0,0)$ and check to see if the point $(\sqrt{R}/2\pi, m)$ lies below or above the ℓth curve. If the point $(\sqrt{R}/2\pi, m)$ lies above the ℓth curve then the mth wave is stable to linear perturbations of the ℓth wave, if it lies below it is linearly unstable to those perturbations.

In addition to the rotating wave solutions both [55] and [41] note the existence of the "Stokes solution" or "monochromatic" time dependent solutions given by

$$U(x,t) = b_m(t)e^{i\Lambda_m x} \tag{3.5.81}$$

where $b_m(t)$ satisfies the following differential equation

$$\frac{d}{dt}b_m(t) = Rb_m(t) - (1+i\nu)\Lambda_m - (1+i\mu)|b_m(t)|^2 b_m(t), \quad m = 0, \pm 1, \cdots. \tag{3.5.82}$$

Provided $R > \Lambda_m$ we may solve (3.5.82) to find $|b_m(t)|^2$

$$|b_m(t)|^2 = \frac{(R-\Lambda_m)|b_m(0)|^2}{(R-\Lambda_m - |b_m(0)|^2)e^{-2(R-\Lambda_m)t} + |b_m(0)^2|}. \tag{3.5.83}$$

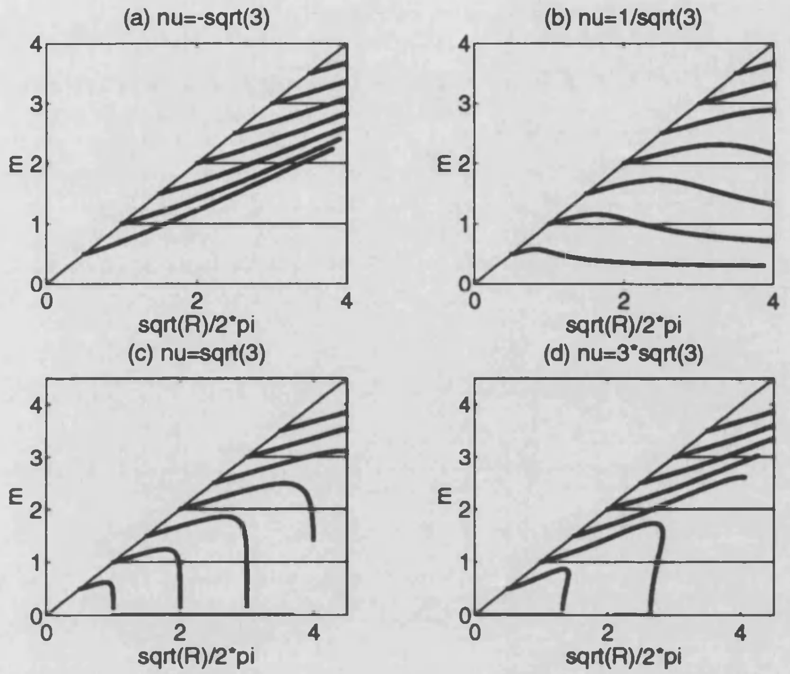

Figure 3.2: Neutral stability curves for $\mu = -\sqrt{3}$, and (a) $\nu = -\sqrt{3}$, (b) $\nu = 1/\sqrt{3}$, (c) $\nu = \sqrt{3}$, (d) $\nu = 3\sqrt{3}$.

From (3.5.83) the rotating waves are contained in the asymptotics of the monochromatic waves. By a suitable choice of $b_m(0)$ we see that the monochromatic solutions give us connections between the rotating waves. Hence the monochromatic waves are exact heteroclinic connections.

In Figure 3.3 we have plotted the spatially homogeneous rotating wave - the dot dashed line - and the monochromatic wave with initial condition close to the trivial solution $U \equiv 0$ which is unstable. We see that the monochromatic wave gives the connection between the trivial solution and the spatially homogeneous wave.

Further details on the bifurcation structure has been provided by Takáč [127] who has proved the existence of invariant 2-tori arising from secondary bifurcations from the branches of rotating waves.

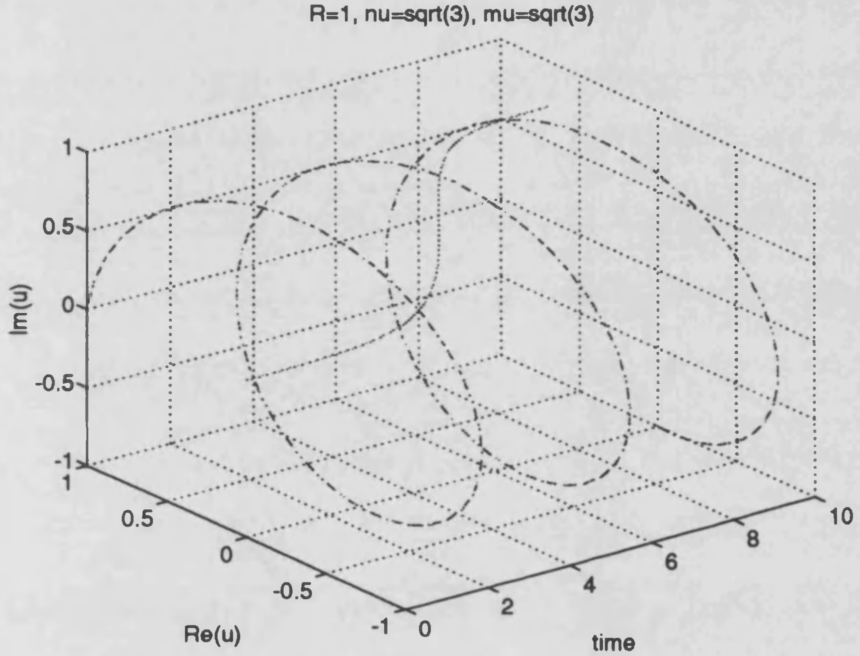

Figure 3.3: Spatially homogeneous rotating wave and monochromatic wave connecting $U \equiv 0$ and the spatially homogeneous wave.

3.6 The Discretizations

In the previous sections we discussed the complex Ginzburg–Landau equation (3.1.1) with periodic boundary conditions and gave an overview of results for this continuous system. Our aim is to investigate finite difference approximations to the Ginzburg–Landau equation and to prove that these discretizations have the same qualitative dynamics as the continuous system. We will prove in Chapters 4 and 5 discrete counterparts to the results outlined earlier in this Chapter. This section introduces the specific discretizations that are treated in the next three chapters.

Consider the interval $[0,1]$ divided into J uniformly distributed mesh points of distance $\Delta x := 1/J$ apart.

We shall let subscripts indicate the spatial mesh point, so that $U_j(\bullet)$ is our approximation to $U(j\Delta x, \bullet)$. Define δ_+ to be the forward difference approximation to the derivative and δ_- to be the backward difference approximation so that

$$\delta_+ U_j = \frac{U_{j+1} - U_j}{\Delta x} \quad ; \quad \delta_- U_j = \frac{U_{j-1} - U_j}{\Delta x}. \tag{3.6.1}$$

From this we may define the standard approximation for second derivatives

$$\delta^2 U_j = \delta_+ \delta_- U_j = \frac{U_{j+1} - 2U_j + U_{j-1}}{\Delta x^2}. \tag{3.6.2}$$

Notation Let $\widetilde{V} \in \ell^2$,

$$\widetilde{V} = (\cdots, V_{-1}, V_0, V_1, \cdots, V_{J-1}, V_J, V_{J+1}, \cdots)^T,$$

be such that

$$\widetilde{V}_k = \widetilde{V}_{k+J} \ \forall \ k \in \mathbb{Z}. \tag{3.6.3}$$

Then $\widetilde{V}$ is said to satisfy periodic boundary conditions.

Henceforward we let

$$V = (V_0, \cdots, V_{J-1}) \in \mathbb{C}^J_{\text{per}}$$

denote $V \in \ell^2$ satisfying (3.6.3) and make free use of the periodicity. Thus, for example,

$$V_{-1} = V_{J-1}, \quad V_J = U_0, \quad V_{J+1} = V_1.$$

Lemma 3.6.1 (Summation by Parts) *Let $U, V \in \mathbb{C}^J_{\text{per}}$ be given by*

$$U = (U_0, \cdots, U_{J-1}), V = (V_0, \cdots, V_{J-1}).$$

Then,

$$\sum_{k=1}^{J-1} U_k \delta_+ V_k = -\sum_{k=1}^{J-1} V_k \delta_+ U_k.$$

Proof Standard summation by parts result (e.g. [95]) gives

$$\sum_{k=1}^{J-1} U_k \delta_+ V_k = U_k V_k|_1^J - \sum_{k=1}^{J-1} V_k \delta_+ U_k. \tag{3.6.4}$$

All that remains is to note that the periodicity implies

$$U_k V_k|_1^J = 0$$

and the proof is complete. □

Let M be the $J \times J$ diagonal matrix with entries Δx down the diagonal :

$$M = \text{diag}(\Delta x, \cdots, \Delta x).$$

In finite element terms M corresponds to a mass matrix constructed through mass lumping. We denote the matrix arising from the forward difference approximation to the derivative by

$$D = \Delta x^{-1} \begin{pmatrix} -1 & 1 & \cdots & 0 \\ 0 & -1 & 1 & \\ & \ddots & \ddots & \\ 1 & & & -1 \end{pmatrix}. \tag{3.6.5}$$

where the non–zero corner element arises from the periodicity. Define A by

$$A := \Delta x^{-1} \begin{pmatrix} 2 & -1 & 0 & \cdots & -1 \\ -1 & 2 & -1 & & 0 \\ & \ddots & \ddots & \ddots & \vdots \\ 0 & & -1 & 2 & -1 \\ -1 & 0 & \cdots & -1 & 2 \end{pmatrix}, \tag{3.6.6}$$

then the matrix

$$M^{-1}A$$

is the matrix representation of the standard finite difference approximation to $-\Delta$ for a problem with periodic boundary conditions arising from (3.6.2).

As in the continuous case (section 3.3) we define a linear operator which allows us to define the mathematical setting for the problem. This new linear operator corresponds to shifting the spectrum to make it positive. In this discrete setting define

$$\tilde{A} = I + M^{-1}A. \tag{3.6.7}$$

All that remains is to introduce notation for the non–linear term. For $V \in \mathbb{C}^J_{\text{per}}$ we introduce the diagonal matrix $G(V)$ with entries V_j on the diagonal. The non–linear term is then written using

$$G(|V|^2) := \begin{pmatrix} |V_0|^2 & & 0 \\ & \ddots & \\ 0 & & |V_{J-1}|^2 \end{pmatrix}. \tag{3.6.8}$$

where it is understood that $|V|^2$ is the element of $\mathbb{C}^J_{\text{per}}$ given by

$$|V|^2 = (|V_0|^2, \cdots, |V_{J-1}|^2)^T.$$

The first discretization is a purely spatial discretization. This reduces the partial differential equation to a finite system of J ordinary differential equations.

Semi–Discrete Problem (SD) :

Find $U(t) = (U_0(t), \cdots, U_{J-1}(t))^T \in \mathbb{C}^J_{\text{per}}$ satisfying

$$\mathbf{SD} \quad \begin{cases} U_t = RU - (1+i\nu)M^{-1}AU - (1+i\mu)G(|U|^2)U \\ \qquad U(0) = U^0 \in \mathbb{C}^J_{\text{per}} \end{cases}. \tag{3.6.9}$$

Equivalently we may write **SD** as

$$U_t + (1+i\nu)\tilde{A}U = F(U) \tag{3.6.10}$$

where

$$F(V) := \tilde{R}V - (1+i\mu)G(|V|^2)V \tag{3.6.11}$$

and $\tilde{R} = (R + (1+i\nu))$. This form of the equation is useful for proving certain analytical results.

Although the semi–discrete problem (3.6.9) is of interest, a discretization in time is required before we get a numerical scheme that we can implement on a computer.

Let $\Delta t \in (0,\infty)$ be the time step and U^n denote the approximation to $U(n\Delta t)$. The time derivative is then approximated by a straightforward one–step finite difference approximation.

From an analytical point of view, the easiest fully discrete problem (i.e. discretized in time and space) to consider is a fully implicit scheme, constructed by applying the backward Euler's method to (3.6.9).

Discrete Fully Implicit Problem (DI) :

Find $U^n = (U_0^n, \cdots, U_{J-1}^n)^T \in \mathbb{C}^J_{\text{per}}$ satisfying

$$\text{DI} \quad \begin{cases} \frac{U^{n+1}-U^n}{\Delta t} = RU^{n+1} - (1+i\nu)M^{-1}AU^{n+1} - (1+i\mu)G(|U^{n+1}|^2)U^{n+1} \\ \qquad U(0) = U^0 \in \mathbb{C}^J_{\text{per}} \end{cases} \tag{3.6.12}$$

or equivalently

$$\frac{U^{n+1}-U^n}{\Delta t} + (1+i\nu)\tilde{A}U^{n+1} = F(U^{n+1}). \tag{3.6.13}$$

$$U(0) = U^0 \in \mathbb{C}^J_{\text{per}}$$

In order to implement this scheme on the computer a non–linear solver such as Newton's method is required. This is because the solution at the $(n+1)^{th}$ time level, U^{n+1}, is given implicitly as a function of the solution at the n^{th} time level U^n. However, based on linear intuition, we can expect this scheme to have good stability properties and to have dynamics close to those of the continuous time systems.

Possibly the easiest numerical scheme to code is the fully explicit scheme:

Discrete Fully Explicit Problem (DFE) :

Find $U^n = (U_0^n, \cdots, U_{J-1}^n) \in \mathbb{C}^J_{\text{per}}$ satisfying

$$\text{DFE} \quad \begin{cases} \frac{U^{n+1}-U^n}{\Delta t} = RU^n - (1+i\nu)M^{-1}AU^n - (1+i\mu)G(|U^n|^2)U^n \\ \qquad U(0) = U^0 \in \mathbb{C}^J_{\text{per}} \end{cases} \tag{3.6.14}$$

since the solution at the $(n+1)^{th}$ time level U^{n+1} is found from U^n by a simple matrix vector multiplication. However we expect this scheme to exhibit dynamics which are possibly far from those exhibited in the continuous case: see for example [53] or [46].

The following two schemes are a combination of explicit and implicit terms on the right hand side.. The first

Discrete Explicit Non–linear Term (DE) :

Find $U^n = (U_0^n, \cdots, U_{J-1}^n)^T \in \mathbb{C}^J_{\text{per}}$ satisfying

$$\textbf{DE} \quad \begin{cases} \frac{U^{n+1}-U^n}{\Delta t} = RU^{n+1} - (1+i\nu)M^{-1}AU^{n+1} - (1+i\mu)G(|U^n|^2)U^n \\ \qquad\qquad U(0) = U^0 \in \mathbb{C}^J_{\text{per}} \end{cases} \tag{3.6.15}$$

has a non–linear term which is fully explicit, and the second

Discrete Explicit-Implicit Non–linear Term (DEI) :

Find $U^n = (U_0^n, \cdots, U_{J-1}^n)^T \in \mathbb{C}^J_{\text{per}}$ satisfying

$$\textbf{DEI} \quad \begin{cases} \frac{U^{n+1}-U^n}{\Delta t} = RU^{n+1} - (1+i\nu)M^{-1}AU^{n+1} - (1+i\mu)G(|U^n|^2)U^{n+1} \\ \qquad\qquad U(0) = U^0 \in \mathbb{C}^J_{\text{per}} \end{cases} \tag{3.6.16}$$

has a mixed non–linear term.

Both the schemes **DE** (3.6.15) and **DEI** (3.6.16) are relatively easy to code as all that is required to find the solution at the $(n+1)^{th}$ time level U^{n+1} from the n^{th} time level U^n is a particular form of Gauss elimination, essentially a tridiagonal solver adapted to handle the two corner elements arising from the periodic boundary conditions. Numerical results for all four of the discrete schemes **DFE** (3.6.14), **DE** (3.6.15), **DEI** (3.6.16) and **DI** (3.6.12) are presented and commented on in section 4.5.

3.7 Discrete Setting and Analysis

This section introduces the mathematical setting for our spatially discrete equations and some discrete Sobolev inequalities are proved for this finite difference setting. Other discrete Sobolev space results may be found, for example, in [100, 124, 136, 137, 142].

We commence by introducing some notation which allows us to deal with the cases J odd and J even simultaneously.

Notation : Let $s \in \mathbb{R}$, then we define

$$s\rceil := \text{ smallest integer } > s \tag{3.7.1}$$

$$s\rfloor := \text{ greatest integer } \leq s. \tag{3.7.2}$$

For example :

$$-4.5\rceil = -4 \qquad -4\rceil = -3$$

and

$$4.5\rfloor = 4 \quad 4\rfloor = 4.$$

The first lemma gives the eigenvalues and eigenvectors for the finite difference approximation to $-\Delta$ and $I - \Delta$.

Lemma 3.7.1 *The linear operator $M^{-1}A$ has eigenvalues*

$$\lambda_k = \frac{4}{\Delta x^2} \sin^2(k\pi\Delta x) \quad k = -J/2\rceil, \cdots, 0, \cdots, J/2\rfloor \tag{3.7.3}$$

and the linear operator $\widetilde{A} = I + M^{-1}A$ has eigenvalues

$$\widetilde{\lambda_k} = 1 + \lambda_k \quad k = -J/2\rceil, \cdots, 0, \cdots, J/2\rfloor. \tag{3.7.4}$$

The corresponding eigenvectors ψ_k for both $M^{-1}A$ and $\widetilde{A}$ are given by

$$\psi_k = \left(1, e^{2\pi ik\Delta x}, \ldots, e^{2\pi ik(J-1)\Delta x}\right) \quad k = -J/2\rceil, \cdots, 0, \cdots, J/2\rfloor. \tag{3.7.5}$$

Proof Standard result - see for example [121]. $\square$

Remark 3.7.1 *The operators $\widetilde{A}$ and $(1 + i\nu)\widetilde{A}$ are sectorial operators. This follows immediately from the finite dimensionality and (as in Theorem 3.2.2) the eigenvalues being bounded away from 0.*

Note that the eigenvalues and eigenvectors for the discrete problem converge to those for the continuous problem as $\Delta x \to 0$. The following lemma bounds the ratio of the continuous eigenvalues (see Theorem 3.3.1) to the discrete eigenvalues (Lemma 3.7.1).

Lemma 3.7.2 *The ratio of discrete to continuous eigenvalues*

$$r(k) = \frac{\lambda_k}{\Lambda_k}, \quad k = -J/2\rceil, \cdots, -1, 1, \cdots, J/2\rfloor$$

satisfies

$$\frac{4}{\pi^2} \le r(k) \le 1 \quad k = -J/2\rceil, \cdots, -1, 1, \cdots, J/2\rfloor \tag{3.7.6}$$

and the ratio

$$\tilde{r}(k) = \frac{\widetilde{\lambda_k}}{\widetilde{\Lambda}_k}, \quad k = -J/2\rceil, \cdots, J/2\rfloor$$

$$\frac{4}{\pi^2} \le \tilde{r}(k) \le 1 \quad k = -J/2\rceil, \cdots, J/2\rfloor. \tag{3.7.7}$$

Proof

First note that

$$\lim_{k\to 0} r(k) = \lim_{k \to 0} \frac{J^2 \sin^2(k\pi\Delta x)}{k^2\pi^2} = 1$$

and

$$r(\frac{J}{2}) = \frac{J^2\sin^2(\pi/2)}{J^2\pi^2/4} = \frac{4}{\pi^2}.$$

It remains to show that $r(k)$ is strictly decreasing. To achieve this we do the natural thing and differentiate :

$$\begin{aligned} r'(k) &= \frac{2J^2\pi\Delta x \sin(k\pi\Delta x)\cos(k\pi\Delta x)}{k^2\pi^2} - \frac{2J^2\sin^2(k\pi\Delta x)}{\pi^2 k^3} \\ &= \frac{2J^2\sin^2(k\pi\Delta x)}{\pi^2k^2}\left\{\frac{k\pi\Delta x - \tan(k\pi\Delta x)}{k\sin(k\pi\Delta x)(\cos(k\pi\Delta x))^{-1}}\right\} \\ &< 0. \end{aligned}$$

Thus we have established (3.7.6). To establish (3.7.7), from (3.7.6)

$$\begin{aligned} \lambda_k \le \Lambda_k &\Longrightarrow 1 + \lambda_k \le 1 + \Lambda_k \\ &\Longrightarrow \frac{\widetilde{\lambda_k}}{\widetilde{\Lambda}_k} \le 1. \end{aligned}$$

Also, since $\frac{4}{\pi^2} < 1$,

$$\frac{4}{\pi^2}\Lambda_k \le \lambda_k \quad \Longrightarrow \frac{4}{\pi^2}(1 + \Lambda_k) \le 1 + \lambda_k$$

$$\frac{4}{\pi^2} \le \frac{\widetilde{\lambda_k}}{\widetilde{\Lambda_k}}$$

whence (3.7.7) is proved. □

Now let us define the following complex spaces which are discrete analogues of the complexified Sobolev spaces of section 3.3,[104] or [128, page 273].

Discrete L^p Spaces : $L^p_{\Delta x}$

Let $U = (U_0, U_1, \ldots, U_{J-1})^T \in \mathbb{C}^J_{\text{per}}$ and $V = (V_0, V_1, \ldots, V_{J-1})^T \in \mathbb{C}^J_{\text{per}}$. The **discrete (complex)** $L^p_{\Delta x}$ **space** on $(0,1)$ is defined to be the normed linear space $\{\mathbb{C}^J, |\bullet|_{L^p_{\Delta x}}\}$ where

$$|V|_{L^p_{\Delta x}} = \begin{cases} \left\{\sum_{j=0}^{J-1} \Delta x |V_j|^p\right\}^{1/p} & \text{for } 1 \le p < \infty \\ \sup_{0 \le j \le J-1} |V_j| & \text{for } p = \infty. \end{cases} \tag{3.7.8}$$

The **discrete** $L^2_{\Delta x}$ **inner product** is defined by

$$\begin{aligned} < U, V > &= U^T M \overline{V} \\ &= \sum_{j=0}^{J-1} \Delta x U_j \overline{V_j}. \end{aligned} \tag{3.7.9}$$

Notes

- The $L^2_{\Delta x}$ inner–product (3.7.10) induces with $p = 2$ the norm of (3.7.8) through $|\bullet|^2_{L^2_{\Delta x}} =< \bullet, \bullet >$.
- We shall seek bounds on the $L^p_{\Delta x}$ norms which are independent of Δx and hence remain valid as $\Delta x \to 0$. Thus $L^p_{\Delta x}$ approximates the space L^p.

Discrete Sobolev Spaces : $H^{2s}_{\Delta x}$

The discrete Sobolev spaces are defined in an analogous manner to section 3.3. Consider the expansion of $V = (V_0, \ldots, V_{J-1})^T \in L^2_{\Delta x}$ as a Fourier series based on the eigenvectors of $\widetilde{A}$ given in Lemma 3.7.1, so that

$$V = \sum_{k=-\lceil J/2 \rceil}^{\lfloor J/2 \rfloor} a_k \psi_k. \tag{3.7.10}$$

Then, for $s > 0$, we define the **Discrete Sobolev Space** H^{2s} as the normed linear space $\left\{\mathbb{C}^{J}_{\text{per}}, \|\bullet\|_{H^{2s}_{\Delta x}}\right\}$ where

$$\|V\|^2_{H^{2s}_{\Delta x}} = |\tilde{A}^s V|^2_{L^2_{\Delta x}} = \sum_{k=-\lceil J/2\rceil}^{\lfloor J/2\rfloor} \tilde{\lambda}_k^{2s}|a_k|^2. \tag{3.7.11}$$

Then:

- For $s = 0$ we simply recover the $L^2_{\Delta x}$ norm.
- For $s = \frac{1}{2}$ we find the space $H^1_{\Delta x}$ which approximates H^1

$$H^1_{\Delta x} = \left\{\mathbb{C}^{J}_{\text{per}}, \|\bullet\|_{H^1_{\Delta x}}\right\} \text{ where } \|V\|^2_{H^1_{\Delta x}} = |\tilde{A}^{1/2} V|^2_{L^2_{\Delta x}} = \sum_{k=-\lceil J/2\rceil}^{\lfloor J/2\rfloor} \tilde{\lambda}_k|a_k|^2. \tag{3.7.12}$$

We can also formulate our discrete $H^m_{\Delta x}$ spaces in terms of discrete approximations to the distributional derivatives.

Define the **discrete Dirichlet inner–product** by

$$\begin{aligned} < U, V >_A &:= U^T A\overline{V} & (3.7.13)\\ &= -\sum_{j=0}^{J-1} \Delta x U_j \frac{\overline{V_{j+1}} - 2\overline{V_j} + \overline{V_{j-1}}}{\Delta x^2} \\ &= -\sum_{j=0}^{J-1} \Delta x U_j \delta^2 \overline{V}_j, & (3.7.14)\end{aligned}$$

where $j = 0$ and $j = J - 1$ are dealt with by the periodicity.

The inner product allows us to define the semi-norm $\|\bullet\|_1$ by

$$\|V\|_1^2 = < V, V >_A = V^T A\overline{V} \geq 0. \tag{3.7.15}$$

An alternative definition for the **discrete** $H^1_{\Delta x}$ **space** is the normed vector space $\{\mathbb{C}^J, \|\bullet\|_{H^1_{\Delta x}}\}$, where

$$\|V\|_{H^1_{\Delta x}} = \left\{|V|^2_{L^2_{\Delta x}} + \|V\|_1^2\right\}^{1/2}; \tag{3.7.16}$$

see Lemma 3.7.4 below.

Note :

- We shall seek to bound the norms $\|\bullet\|_{H^{2s}_{\Delta x}}$ independently of Δx so that the bounds continue to hold as $\Delta x \to 0$. Thus $H^{2s}_{\Delta x}$ approximates the space H^{2s}.

The next few results concern the relation between the continuous spaces H^{2s} and the discrete spaces $H^{2s}_{\Delta x}$. We relate the discrete space $H^1_{\Delta x}$ and the conventional Sobolev space H^1 in the following Theorem, first however a definition.

Definition 3.7.1 Let $\mathbf{V} \subset H^1$ be the space of piecewise–linear functions and let $\{\phi_j\}$ be the standard basis of "hat" functions given by

$$\phi_k(x) = \begin{cases} (x - x_{k-1})/\Delta x & \text{for } x_{k-1} \leq x < x_k \\ (x_{k+1} - x)/\Delta x & \text{for } x_k \leq x < x_{k+1} \\ 0 & \text{otherwise;} \end{cases}$$

with

$$\phi_0(x) = \begin{cases} (x_1 - x)/\Delta x & \text{for } x_0 \leq x < x_1 \\ 0 & \text{otherwise.} \end{cases},$$

$$\phi_{J-1}(x)) = \begin{cases} (x - x_{J-2})/\Delta x & \text{for } x_{J-2} \leq x < x_{J-1} \\ 0 & \text{otherwise.} \end{cases}$$

and $x_j = j\Delta x$. Then we define the *prolongation of $H^1_{\Delta x}$ into H^1*,

$$P_L : H^1_{\Delta x} \rightarrow \mathbf{V} \subset H^1,$$

for $V \in \mathbb{C}^J_{\text{per}}$ by

$$P_L V = \sum_{j=0}^{J-1} V_j \phi_j.$$

Theorem 3.7.1 *Let $\mathbf{V} \subset H^1$ be the space of piecewise–linear functions, let $V \in H^1_{\Delta x}$ and define $v \in \mathbf{V}$ by*

$$v := P_L V,$$

where P_L is the prolongation defined in Definition 3.7.1. Then $\exists$ a constant $C > 0$ independent of Δx such that

$$\frac{1}{C}||v||_{H^1} \leq ||V||_{H^1_{\Delta x}} \leq C||v||_{H^1}.$$

Hence we have a norm equivalence between $H^1_{\Delta x}$ and H^1.

Proof We recall that this proof is set in $p = 1$ dimension.

Recall that $||v||_{H^1} = \left\{|v|^2_{L^2} + |A_0^{1/2}v|^2_{L^2}\right\}^{\frac{1}{2}}$ and let $\{\phi_j\}$ denote the standard basis for $\mathbf{V}$. Now by standard finite element analysis,

$$||A_0^{1/2}v||^2_{L^2} =< A_0^{1/2}v, A_0^{1/2}v >= V^T K \overline{V}$$

where K is the so called *stiffness matrix* given by $K_{i,j} =< A\phi_i, \phi_j >$. Evaluating the inner products we find in 1 dimension that

$$K = A.$$

The proof is completed by noting another standard result (for example Hackbusch [60, Theorem 8.8.1]) that $\exists$ constant C independent of Δx such that

$$\frac{1}{C}|v|_{L^2} \leq |V|_{L^2_{\Delta x}} \leq C|v|_{L^2}.$$

This is proved by looking at the mass matrix. Estimates for the constant C may be found in [131]. □

A straightforward way to relate the conventional Sobolev spaces H^{2s} and the discrete spaces $H^{2s}_{\Delta x}$ is given in the following Lemma, but first we require a definition.

Definition 3.7.2 Let $s \geq \frac{1}{2}$ and define $P_{\Delta x} : H^{2s} \to H^{2s}_{\Delta x}$ to be the operator which evaluates the continuous function $V(x) \in H^{2s}$ at the grid points. Thus

$$P_{\Delta x}V(x) = (V(0), V(\Delta x), \ldots, V((J-1)\Delta x))^T.$$

Let $W \subset H^{2s}$, then

$$P_{\Delta x}W = \bigcup_{w \in W} P_{\Delta x}w.$$

Note : Since we are in 1 spatial dimension the projection $P_{\Delta x}$ is well defined by the Sobolev embedding theorems (e.g. [104, p208]) for $s \geq 1/2$

$$V \in H^{2s}, \Longrightarrow V \in L^{\infty} \Longrightarrow V \in C^0.$$

Indeed Strikwerda [124, Sec 10.1] shows that in one space dimension this evaluation on the grid is well defined for $V \in H^{2s}$, $s > 1/4$.

The following lemma shows that if we project a function in H^{2s} onto the grid, then the $H^{2s}_{\Delta x}$ norm of the resulting vector is bounded.

Lemma 3.7.3 *Let $v \in H^{2s}$, $s \geq \frac{1}{2}$ have Fourier expansion*

$$v = \sum_{-\infty}^{\infty} a_k e^{2\pi i k x}.$$

Then,

$$||P_{\Delta x}v||_{H^{2s}_{\Delta x}} \leq 2||v||_{H^{2s}}.$$

Proof Note that from definition of $P_{\Delta x}$

$$\begin{aligned}(P_{\Delta x}v)_j &= \sum_{-\infty}^{\infty} a_k e^{2\pi ikj\Delta x} \\ &= \sum_{k=-\lceil J/2 \rceil}^{\lfloor J/2 \rfloor} b_k e^{2\pi ikj\Delta x},\end{aligned}$$

where

$$b_k = a_k + a_{k+J} + a_{k-J} + a_{k+2J} + \cdots.$$

Now we have that

$$|b_k|^2 \leq 2 \sum_{m=-\infty}^{\infty} |a_{k+mJ}|^2,$$

where the sum converges since $v \in L^2$. Further, by the definition of the H^{2s} norm we get that

$$\begin{aligned}\|P_{\Delta x}v\|^2_{H^{2s}_{\Delta x}} &= \sum_{k=-\lceil J/2 \rceil}^{\lfloor J/2 \rfloor} \widetilde{\lambda}_k^{2s} |b_k|^2 \\ &\leq 2\sum_{-\infty}^{\infty} \widetilde{\lambda}_k^{2s} |a_k|^2. \qquad (3.7.17)\end{aligned}$$

If we can show that $\widetilde{\lambda_k} \leq \widetilde{\Lambda}_k \ \forall\ k \in \mathbb{Z}$ then our proof is complete.

• For $k \in [-\lceil J/2 \rceil, \lfloor J/2 \rfloor]$ we see by Lemma 3.7.2 that $\widetilde{\lambda}_k \leq \widetilde{\Lambda}_k$.

• For $k \in (-\infty, -\lceil J/2 \rceil) \bigcup (\lfloor J/2 \rfloor, \infty)$ we have that $\widetilde{\lambda}_k$ is periodic and attains its maximum at $k = J/2 + \eta\pi, \eta \in \mathbb{Z}$.

• For $k \in (-\infty, -\lfloor J/2 \rfloor) \bigcup (\lfloor J/2 \rfloor, \infty)$ we have that $\widetilde{\Lambda}_k$ is strictly increasing and that $\widetilde{\lambda}_k \leq \widetilde{\Lambda}_k$.

Hence

$$\widetilde{\lambda}_k \leq \widetilde{\Lambda}_k, \quad \forall\ k \in (-\infty, \infty).$$

We apply this to (3.7.17) to get

$$\|P_{\Delta x}v\|^2_{H^{2s}_{\Delta x}} \leq 2\sum_{-\infty}^{\infty} \widetilde{\lambda}_k^{2s} |a_k|^2 \leq 2\sum_{-\infty}^{\infty} \widetilde{\Lambda}_k^{2s} |a_k|^2 = 2\|v\|^2_{H^{2s}},$$

and hence the Lemma is proved. □

We now turn our attention to proving some results about the discrete spaces. The next couple of lemmas are results on norm equivalence.

Lemma 3.7.4 *The two definitions of the $H^1_{\Delta x}$ spaces given by* (3.7.12) *and* (3.7.16) *are equivalent and, in fact,*

$$\left\{|V|^2_{L^2_{\Delta x}} + \|V\|^2_1\right\}^{1/2} = |\widetilde{A}^{\frac{1}{2}} V|_{L^2_{\Delta x}}.$$

Proof Recall $\widetilde{A} = I + M^{-1}A$; then using the symmetry of $M^{-1}A$ and that $M, M^{-1}A$ and $(I + M^{-1}A)^{1/2}$ commute

$$\begin{aligned}
|[I + M^{-1}A]^{\frac{1}{2}} V|^2_{L^2_{\Delta x}} &= \left([I + M^{-1}A]^{\frac{1}{2}} V\right)^T M \left([I + M^{-1}A]^{\frac{1}{2}} \overline{V}\right) \\
&= V^T \left([I + M^{-1}A]^{\frac{1}{2}}\right)^T M [I + M^{-1}A]^{\frac{1}{2}} \overline{V} \\
&= V^T M [I + M^{-1}A] \overline{V} \\
&= V^T M \overline{V} + V^T A \overline{V}
\end{aligned} \tag{3.7.18}$$

and we have proved the stated equivalence. □

Lemma 3.7.5 *The $H^2_{\Delta x}$ space as defined by* (3.7.11) *is norm equivalent to the space defined by discrete approximations to the appropriate distributional derivatives. Specifically we have*

$$\frac{1}{2}(|V|^2_{L^2_{\Delta x}} + \|V\|^2_1 + |M^{-1}AV|^2_{L^2_{\Delta x}}) \leq |\widetilde{A}V|^2_{L^2_{\Delta x}} \leq 2(|V|^2_{L^2_{\Delta x}} + \|V\|^2_1 + |M^{-1}AV|^2_{L^2_{\Delta x}}).$$

Furthermore we have the inequality

$$\|V\|^2_1 + |M^{-1}AV|^2_{L^2_{\Delta x}} \leq 2\{|V|^2_{L^2_{\Delta x}} + |M^{-1}AV|^2_{L^2_{\Delta x}}\}.$$

Proof Recall again that $\widetilde{A} = I + M^{-1}A$. Use the symmetry of $M^{-1}A$ and the fact that $M, M^{-1}A$ and $I + M^{-1}A$ commute to get

$$\begin{aligned}
|\widetilde{A}V|^2_{L^2_{\Delta x}} &= ([I + M^{-1}A]V)^T M [I + M^{-1}A] V \\
&= V^T [I + M^{-1}A]^T M [I + M^{-1}A] V \\
&= V^T M V + V^T A V + V^T A V + V^T (M^{-1}A)^T M (M^{-1}AV) \\
&= |V|^2_{L^2_{\Delta x}} + 2\|V\|^2_1 + |M^{-1}AV|^2_{L^2_{\Delta x}}.
\end{aligned}$$

Hence we have proved the equivalence.

To establish the final inequality note that $\forall\ s \geq 0$

$$s + s^2 \leq 1 + 2s + s^2 - s \leq (1+s)^2 \leq 2(1+s^2). \qquad (3.7.19)$$

Now by (3.7.10)

$$\|V\|_1^2 + |M^{-1}AV|^2_{L^2_{\Delta x}} = \sum_{k=-\lceil J/2\rceil}^{\lfloor J/2\rfloor} |a_k|^2 \left(\lambda_k + \lambda_k^2\right). \qquad (3.7.20)$$

The desired result follows by applying the inequality (3.7.19) to the eigenvalues $\widetilde{\lambda}_k$ to get

$$\begin{aligned} \|V\|_1^2 + |M^{-1}AV|^2_{L^2_{\Delta x}} &\leq 2 \sum_{k=-\lceil J/2\rceil}^{\lfloor J/2\rfloor} |a_k|^2 \left(1 + \lambda^2\right) \\ &= 2|V|^2_{L^2_{\Delta x}} + |M^{-1}AV|^2_{L^2_{\Delta x}} \qquad \square \end{aligned}$$

It should be noted in the previous lemma that the norm equivalence was independent of the spatial mesh Δx.

Lemma 3.7.6 *The following two equalities hold :*

$$i) \qquad |DV|^2_{L^2_{\Delta x}} = \|V\|_1^2;$$

$$ii) \qquad \|DV\|_1^2 = |M^{-1}AV|^2_{L^2_{\Delta x}}.$$

Proof By equation (3.7.14)

$$\|V\|_1^2 = \sum_{j=0}^{J-1} \Delta x V_j \frac{\overline{V_{j+1}} - 2\overline{V_j} + \overline{V_{j-1}}}{\Delta x^2}.$$

Now by summation by parts (Lemma 3.6.1) and the periodic boundary conditions we see that $|DV|^2_{L^2_{\Delta x}} = \|V\|_1^2$ and we have proved $i)$.
The proof of $ii)$ is just as straightforward. By (3.6.5) we have that

$$(DV)_j = \frac{(V_{j+1} - V_j)}{\Delta x} = \delta_+ V_j,$$

so that by definition (3.7.14)

$$\|DV\|_1^2 = -\sum_{j=0}^{J-1} \Delta x (\delta_+ V_j) \delta^2 (\delta_+ \overline{V_j}).$$

By summation by parts we find

$$\|DV\|_1^2 = \sum_{j=0}^{J-1} \Delta x \, |\delta_+(V_{j+1} - V_j)|^2 .$$

Now note

$$(V_{j+2} - V_{j+1}) - (V_{j+1} - V_j) = V_{j+2} - 2V_{j+1} - V_j,$$

expand $\delta_+(V_{j+1} - V_j)$ and use summation by parts and the boundary conditions,

$$\begin{aligned} \|DV\|_1^2 &= \sum_{j=0}^{J-1} \Delta x \left| \frac{(V_{j+2} - V_{j+1}) - (V_{j+1} - V_j)}{\Delta x} \right|^2 \\ &= \sum_{j=0}^{J-1} \Delta x \left| \frac{V_{j+2} - 2V_{j+1} + V_j}{\Delta x^2} \right|^2 . \end{aligned}$$

Now by the periodicity of the boundary conditions:

$$\begin{aligned} \|DV\|_1^2 &= \sum_{j=0}^{J-1} \Delta x \left| \frac{V_{j+1} - 2V_j + V_{j-1}}{\Delta x^2} \right|^2 \\ &= < M^{-1}AV, M^{-1}AV > . \qquad \square \end{aligned}$$

We now present some lemmas which are discrete versions of well known continuous results.

Lemma 3.7.7 *Given any $V \in L^p_{\Delta x}$, $1 \le p < \infty$,*

$$|V|^p_{L^p_{\Delta x}} \le |V|^p_{L^{2p}_{\Delta x}}$$

Proof This is an application of Hölder's inequality:

$$\begin{aligned} |V|^p_{L^p_{\Delta x}} &= \sum_{j=0}^{J-1} \Delta x |V_j|^p \\ &\le \left\{ \sum_{j=0}^{J-1} \Delta x \right\}^{1/2} \left\{ \sum_{j=0}^{J-1} \Delta x |V_j|^{2p} \right\}^{1/2} \\ &= |V|^p_{L^{2p}_{\Delta x}} . \qquad \square \end{aligned}$$

Lemma 3.7.8 (Bounds on $L^\infty_{\Delta x}$) *The discrete space $H^1_{\Delta x}$ is embedded in the discrete space $L^\infty_{\Delta x}$: that is, $\forall\ V \in H^1_{\Delta x}$*

$$i) \quad |V|^2_{L^\infty_{\Delta x}} \le 3\|V\|^2_{H^1_{\Delta x}} .$$

Also, for any $\epsilon > 0$, we have

$$ii)\ \ |V|^2_{L^\infty_{\Delta x}} \leq (1+\frac{2}{\epsilon^2})|V|^2_{L^2_{\Delta x}} + \frac{1}{2}\epsilon^2\|V\|_1^2.$$

Finally,

$$iii)\ \ |V|^2_{L^\infty_{\Delta x}} \leq |V|^2_{L^2_{\Delta x}} + 2|V|_{L^2_{\Delta x}}\|V\|_1.$$

Proof

i) It is easily verified that for any j

$$\begin{aligned}\delta_+(V_j\overline{V_j}) &= \frac{(V_{j+1}\overline{V_{j+1}} - V_j\overline{V_j})}{\Delta x}\\ &= V_j\delta_+\overline{V_j} + \overline{V_{j+1}}\delta_+V_j.\end{aligned}$$

Summing over j from k_0 to $k-1$ we get

$$|V_k|^2 - |V_{k_0}|^2 = \sum_{j=k_0}^{k-1}\Delta x(V_j\delta_+\overline{V}_j + \overline{V}_{j+1}\delta_+V_j),$$

and now taking the real part yields

$$\begin{aligned}|V_k|^2 &= |V_{k_0}|^2 + \frac{1}{2}\sum_{j=k_0}^{k-1}\Delta x\left((V_j+V_{j+1})\delta_+\overline{V_j} + (\overline{V_j}+\overline{V_{j+1}})\delta_+V_j\right)\\ &\leq |V_{k_0}|^2 + \sum_{j=k_0}^{k-1}\Delta x\left(|V_j+V_{j+1}|\,|\delta_+V_j|\right) \qquad (3.7.21)\end{aligned}$$

To get the embedding we use the generalized Cauchy–Schwarz inequality ($2ab \leq \epsilon^2a^2 + \frac{1}{\epsilon^2}b^2$), complete the square and use the periodic boundary conditions:

$$\begin{aligned}|V_k|^2 &\leq |V_{k_0}|^2 + \frac{1}{2}\sum_{j=k_0}^{k-1}\Delta x\left(\frac{1}{\epsilon^2}|V_j+V_{j+1}|^2 + \epsilon^2|\delta_+V_j|^2\right)\\ &\leq |V_{k_0}|^2 + \frac{1}{2}\sum_{j=0}^{J-1}\Delta x\left(\frac{4}{\epsilon^2}|V_j|^2 + \epsilon^2|\delta_+V_j|^2\right) \qquad (3.7.22)\\ &\leq |V_{k_0}|^2 + 2\sum_{j=0}^{J-1}\Delta x\left(\frac{1}{\epsilon^2}|V_j|^2 + \epsilon^2|\delta_+V_j|^2\right). \qquad (3.7.23)\end{aligned}$$

If we now let $\epsilon^2 = 1$ in (3.7.23) and apply Lemma 3.7.6 we find

$$|V_k|^2 \leq |V_{k_0}|^2 + 2\left(|V|^2_{L^2_{\Delta x}} + \|V\|_1^2\right).$$

Now if we sum over k_0, since $\sum_{k_0=1}^{J-1}\Delta x = 1$ we get:

$$|V_k|^2 \leq |V|^2_{L^2_{\Delta x}} + 2\left(|V|^2_{L^2_{\Delta x}} + \|V\|_1^2\right).$$

Since this is true for any k, by (3.7.16)

$$|V|^2_{L^\infty_{\Delta x}} \leq 3\|V\|^2_{H^1_{\Delta x}}$$

and we have proved part *i*) of our Lemma.

Inequality *ii*) follows from (3.7.22) by summing over k_0.

iii) is found by applying Cauchy–Schwarz to (3.7.21) :

$$\begin{aligned}
|V_k|^2 &\leq |V_{k_0}|^2 + \left\{\sum_{j=0}^{J-1} \Delta x |V_j + V_{j+1}|^2\right\}^{1/2} \left\{\sum_{j=0}^{J-1} \Delta x \, |\delta_+ V_j|^2\right\}^{1/2} \\
&\leq |V_{k_0}|^2 + \left\{2\sum_{j=0}^{J-1} \Delta x \, (|V_j|^2 + |V_{j+1}|^2)\right\}^{1/2} \left\{\sum_{j=0}^{J-1} \Delta x \, |\delta_+ V_j|^2\right\}^{1/2} \\
&\leq |V_{k_0}|^2 + \sqrt{2}\left\{\sum_{j=0}^{J-1} \Delta x |V_j|^2 + \sum_{j=0}^{J-1} \Delta x |V_{j+1}|^2\right\}^{1/2} \left\{\sum_{j=0}^{J-1} \Delta x \, |\delta_+ V_j|^2\right\}^{1/2} .
\end{aligned}$$

Thus, using the periodic boundary conditions,

$$|V_k|^2 = |V_{k_0}|^2 + 2|V|_{L^2_{\Delta x}} \|V\|_1.$$

Once again summing over k_0 and noticing that the choice of k was arbitrary we obtain the last inequality.□

Lemma 3.7.9 (Gagliardo–Nirenberg) *For $1 \leq q \leq p < \infty$ we have the following bounds on the $L^p_{\Delta x}$ norm,*

$$|V|^p_{L^p_{\Delta x}} \leq |V|^{p-q}_{L^\infty_{\Delta x}} |V|^q_{L^q_{\Delta x}}, \tag{3.7.24}$$

$$|V|^p_{L^p_{\Delta x}} \leq 3^{\frac{p-q}{2}} \|V\|^{p-q}_{H^1_{\Delta x}} |V|^q_{L^q_{\Delta x}}, \tag{3.7.25}$$

and finally

$$|V|^p_{L^p_{\Delta x}} \leq \left\{|V|^2_{L^2_{\Delta x}} + 2|V|_{L^2_{\Delta x}} \|V\|_1\right\}^{(p-q)/2} |V|^q_{L^q_{\Delta x}} \tag{3.7.26}$$

Proof Use the definition of the $L^p_{\Delta x}$ norm (3.7.8) to get:

$$|V|^p_{L^p_{\Delta x}} = \sum_{j=0}^{J-1} \Delta x |V_j|^{p-q} |V_j|^q \leq \sup_{0 \leq j \leq J-1} |V_j|^{p-q} \sum_{j=0}^{J-1} \Delta x |V_j|^q = |V|^{p-q}_{L^\infty_{\Delta x}} |V|^q_{L^q_{\Delta x}},$$

hence we have (3.7.24).

To get (3.7.25) we simply apply Lemma 3.7.8 *i)* to bound the $L^\infty_{\Delta x}$ norm.

The final inequality is found by applying Lemma 3.7.8 *iii)* to $L^\infty_{\Delta x}$ norm. □

Note Our proof of the Gagliardo–Nirenberg inequality (Lemma 3.7.9) relies on the *one* dimensional result of Lemma 3.7.8. There are higher dimensional discrete Gagliardo–Nirenberg inequalities in the literature which could be applied (see for example [100]).

Lemma 3.7.10 *We have the following interpolation inequality:*

$$\|V\|_1^2 \le |V|_{L^2_{\Delta x}} |M^{-1}AV|_{L^2_{\Delta x}}$$

Proof Simply apply Cauchy-Schwarz to the definition of $\|V\|_1^2$ from (3.7.14)

$$\|V\|_1^2 = \sum_{j=0}^{J-1} \Delta x V_j \delta^2 \overline{V_j} \le \left\{ \sum_{j=0}^{J-1} \Delta x |V_j|^2 \right\}^{1/2} \left\{ \sum_{j=0}^{J-1} \Delta x |\delta^2 V_j|^2 \right\}^{1/2}.$$

□

Lemma 3.7.11 *The $L^4_{\Delta x}$ norm of DV satisfies*

$$|DV|^4_{L^4_{\Delta x}} \le 6 \left\{ |V|^2_{L^2_{\Delta x}} + |M^{-1}AV|^2_{L^2_{\Delta x}} \right\} \|V\|_1^2.$$

Proof If we apply Lemma 3.7.9, inequality (3.7.25) to DV we get:

$$\begin{aligned} |DV|^4_{L^4_{\Delta x}} &\le 3\|DV\|^2_{H^1_{\Delta x}} |DV|^2_{L^2_{\Delta x}} \\ &= 3\left\{ \|DV\|_1^2 + |DV|^2_{L^2_{\Delta x}} \right\} |DV|^2_{L^2_{\Delta x}}, \end{aligned}$$

so combining this and Lemma 3.7.6 gives:

$$|DV|^4_{L^4_{\Delta x}} \le 3\{\|V\|_1^2 + |M^{-1}AV|^2_{L^2_{\Delta x}}\}\|V\|_1^2.$$

By Lemma 3.7.5 the term in brackets is norm equivalent to the natural norm on $H^2_{\Delta x}$. So

$$|DV|^4_{L^4_{\Delta x}} \le 6\{|V|^2_{L^2_{\Delta x}} + |M^{-1}AV|^2_{L^2_{\Delta x}}\}\|V\|_1^2. \qquad \square$$

The next couple of lemmas are inverse inequalities, that is inequalities which rely on the finite dimensionality and do *not* hold as the spatial step $\Delta x \to 0$.

Lemma 3.7.12 *Given any* $V \in L^2_{\Delta x}$ *we have the following inverse inequality*

$$|V|_{L^\infty_{\Delta x}} \leq \Delta x^{-1/2} |V|_{L^2_{\Delta x}}.$$

Proof Note that

$$|V|^2_{L^2_{\Delta x}} = \sum_{j=0}^{J-1} \Delta x |V_j|^2 \geq \Delta x \sup_{0 \leq j \leq J-1} |V_j|^2,$$

from which the lemma is immediate.□

Lemma 3.7.13 *Given any* $V \in L^2_{\Delta x}$*, we have the following inverse inequalities*

$$\|V\|_1^2 \leq \frac{4}{\Delta x^2} |V|^2_{L^2_{\Delta x}}$$

and

$$|M^{-1}AV|^2_{L^2_{\Delta x}} \leq \frac{4}{\Delta x^2} \|V\|_1^2.$$

Proof Consider the Fourier expansion of V given by (3.7.10). Then,

$$|V|^2_{L^2_{\Delta x}} = \sum_{k=-\lceil J/2 \rceil}^{\lfloor J/2 \rfloor} |a_k|^2 \; ; \; \|V\|_1^2 = \sum_{k=-\lceil J/2 \rceil}^{\lfloor J/2 \rfloor} \lambda_k |a_k|^2 \; ; \; |M^{-1}AV|^2_{L^2_{\Delta x}} = \sum_{k=-\lceil J/2 \rceil}^{\lfloor J/2 \rfloor} \lambda_k^2 |a_k|^2.$$

All that remains to do is to note that

$$\lambda_k = \frac{4}{\Delta x^2} \sin^2(k\pi \Delta x) \leq \frac{4}{\Delta x^2}$$

to get

$$\|V\|_1^2 \leq \frac{4}{\Delta x^2} |V|^2_{L^2_{\Delta x}} \quad \text{and} \quad |M^{-1}AV|^2_{L^2_{\Delta x}} \leq \frac{4}{\Delta x^2} \|V\|_1^2. \qquad \square$$

Discrete Gevrey Class : $G_{\tau,\Delta x}$

In Section 3.4.2 the concept of Gevrey class and regularity was introduced for the continuous Ginzburg–Landau equation. Here we introduce the notion of discrete Gevrey class and regularity.

Given a function $V \in \mathbb{C}^J_{\text{per}}$ with Fourier expansion given by (3.7.10) we define the **discrete Gevrey class of regularity** τ, $G_{\tau,\Delta x}$, to be the normed linear space $\left\{\mathbb{C}^J_{\text{per}}, |\widetilde{A}^s e^{\tau \widetilde{A}^s} \bullet |^2_{L^2_{\Delta x}}\right\}$ where

$$|\widetilde{A}^s e^{\tau \widetilde{A}^s} V|^2_{L^2_{\Delta x}} = \sum_{k=-\lceil J/2 \rceil}^{\lfloor J/2 \rfloor} \widetilde{\lambda}_k^{2s} e^{2\tau \widetilde{\lambda}_k^s} |a_k|^2 \leq C < \infty \qquad (3.7.27)$$

and $\widetilde{\lambda}_k$ is the k^{th} eigenvalue of $\widetilde{A}$ given in Lemma 3.7.1 .

Note We shall only use the case $s = \frac{1}{2}$.

Our first lemma is a discrete version of the continuous result that $H^{2s} \subset G_\tau$ for any $s > 0$.

Lemma 3.7.14 *Let $V \in \mathbb{C}^J_{\rm per}$ and suppose $\exists$ constants C and $\tau > 0$ such that*

$$|\widetilde{A}^{1/2} e^{\tau \widetilde{A}^{1/2}} V|^2_{L^2_{\Delta x}} \leq C. \tag{3.7.28}$$

Then for any $\alpha > 0$,

$$||V||^2_{H^{2\alpha}_{\Delta x}} = |\widetilde{A}^{\alpha} V|^2_{L^2_{\Delta x}} \leq C.$$

Proof First we establish that $\forall \alpha, \tau > 0$ and $x > 1$,

$$f(x) := x^{2\alpha} - x e^{2\tau x^{1/2}} < 0.$$

Now $f(1) < 0$ and $\lim_{x\to\infty} f(x) = -\infty$, and we differentiate to show f is strictly decreasing

$$f'(x) = 2\alpha x^{2\alpha - 1} - e^{2\tau x^{1/2}} \left\{1 + \tau x^{1/2}\right\} \leq 0.$$

We now apply this to bound $\lambda_k^{2\alpha}$ by $\lambda_k e^{2\tau \lambda_k^{1/2}}$ in

$$|\widetilde{A}^{\alpha} V|^2_{L^2_{\Delta x}} = \sum_{k=-\lceil J/2 \rceil}^{\lfloor J/2 \rfloor} \lambda_k^{2\alpha} |a_k|^2 \leq \sum_{k=-\lceil J/2 \rceil}^{\lfloor J/2 \rfloor} \lambda_k e^{2\tau \lambda_k^{1/2}} |a_k|^2 \leq C,$$

and the lemma is proved. □

This next lemma is the discrete version of the continuous result that for $\sigma < \tau$, $G_\tau \subset G_\sigma$.

Lemma 3.7.15 *Let $V \in \mathbb{C}^J_{\rm per}$ and suppose $\exists$ constants C, $\tau > 0$ such that*

$$|\widetilde{A}^s e^{\tau \widetilde{A}^s} V|^2_{L^2_{\Delta x}} \leq C < \infty.$$

Then $\forall \sigma \in (0, \tau]$ we have

$$|\widetilde{A}^s e^{\sigma \widetilde{A}^s} V|^2_{L^2_{\Delta x}} \leq |\widetilde{A}^s e^{\tau \widetilde{A}^s} V|^2_{L^2_{\Delta x}} \leq C.$$

Proof Straightforward. □

We shall now prove that given a $U \in G_{\tau,\Delta x}$, which has Gevrey norm bounded independently of Δx, we can find a corresponding V in a continuous Gevrey class which equals U on the grid. To achieve this recall definition 3.7.2 of the projection $P_{\Delta x}$.

Lemma 3.7.16 *Suppose we are given $U \in \mathbb{C}^J_{\text{per}}$ and constants $C, \tau > 0$ so that U satisfies*

$$|\widetilde{A}^{1/2} e^{\tau \widetilde{A}^{1/2}} U|^2_{L^2_{\Delta x}} \leq C < \infty.$$

Then there exists V,

$$V \in G_\sigma, \quad \sigma \in (0, \frac{2\tau}{\pi})$$

such that

$$|\widetilde{A}_0^{1/2} e^{\sigma \widetilde{A}_0^{1/2}} V|^2_{L^2} \leq C \text{ and } P_{\Delta x} V = U.$$

Furthermore $V \in H^\alpha$, $\forall \alpha \geq 0$.

Proof Consider the Fourier expansion of U :

$$U = \sum_{k=-\lceil J/2 \rceil}^{\lfloor J/2 \rfloor} a_k \psi_k \in L^2_{\Delta x}$$

and define

$$V(x) := \sum_{k=-\lceil J/2 \rceil}^{\lfloor J/2 \rfloor} a_k e^{2\pi i k x}. \tag{3.7.29}$$

Clearly

$$P_{\Delta x} V(x) = U$$

and

$$\begin{aligned} |V|^2_{L^2} &= \int_0^1 \left(\sum_{k=-\lceil J/2 \rceil}^{\lfloor J/2 \rfloor} a_k e^{2\pi i k x} \right) \left(\sum_{k=-\lceil J/2 \rceil}^{\lfloor J/2 \rfloor} \overline{a_k} e^{-2\pi i k x} \right) dx \\ &= \sum_{k=-\lceil J/2 \rceil}^{\lfloor J/2 \rfloor} |a_k|^2 < \infty. \end{aligned}$$

So certainly V is in L^2.

In order to prove V is Gevrey of regularity σ consider

$$\sum_{k=-\lceil J/2 \rceil}^{\lfloor J/2 \rfloor} \widetilde{\Lambda}_k e^{2(\widetilde{\Lambda}_k^{1/2})\sigma} |a_k|^2 = \sum_{k=-\lceil J/2 \rceil, k \neq 0}^{\lfloor J/2 \rfloor} \widetilde{\lambda}_k e^{2\widetilde{\lambda}_k^{1/2}\tau} |a_k|^2 \left\{ \frac{\widetilde{\Lambda}_k}{\widetilde{\lambda}_k} e^{-2\widetilde{\lambda}_k^{1/2}\tau + 2(\widetilde{\Lambda}_k^{1/2})\sigma} \right\}.$$

Re–writing the righthand side so we can apply Lemma 3.7.2

$$
\begin{aligned}
\sum_{k=-J/2\rceil}^{J/2\rfloor} \widetilde{\Lambda}_k e^{2(\widetilde{\Lambda}_k^{1/2})\sigma}|a_k|^2 &= \sum_{k=-J/2\rceil,k\neq 0}^{J/2\rfloor} \widetilde{\lambda}_k e^{2\widetilde{\lambda}_k^{1/2}\tau}|a_k|^2 \left\{\frac{1}{r(k)} e^{2(-r(k)^{1/2}\tau+\sigma)\widetilde{\Lambda}_k^{1/2}}\right\} \\
&\leq \sum_{k=-J/2\rceil,k\neq 0}^{J/2\rfloor} \widetilde{\lambda}_k e^{2\widetilde{\lambda}_k^{1/2}\tau}|a_k|^2 \left\{\frac{\pi^2}{4} e^{2(-\frac{2\tau}{\pi}+\sigma)\widetilde{\Lambda}_k^{1/2}}\right\} \\
&\leq \frac{\pi^2}{4} \sum_{k=-J/2\rceil,k\neq 0}^{J/2\rfloor} \widetilde{\lambda}_k e^{2\widetilde{\lambda}_k^{1/2}\tau}|a_k|^2
\end{aligned}
$$

as $\sigma \in (0, 2\tau/\pi)$. Since (3.7.28) holds we have a uniform bound on the right–hand side. Thus $V(t_0) \in G_\sigma$.

The rest is an easy consequence of the Gevrey regularity and follows immediately from Lemma 3.3.4.□

Smoothing Property

Recall that in Theorem 3.2.9 we presented a theorem which stated that the action of a linear semi–group generated by a sectorial operator was smoothing – in the same way as the action of the Laplacian smoothes solutions to the heat equation. We now consider the smoothing action of the linear semi–groups generated by $\widetilde{A}$ in the semi–discrete case.

Theorem 3.7.2 *Consider the linear homogeneous problem given by*

$$u_t = -(1+i\nu)\widetilde{A}u.$$

Then linear operator $-(1+i\nu)\widetilde{A}$ is the infinitesimal generator of an analytic semi-group $E_{\Delta x}(t)$ defined by

$$E_{\Delta x}(t) := e^{-(1+i\nu)\widetilde{A}t}.$$

Proof Since by Remark 3.7.1 we have that $(1+i\nu)\widetilde{A}$ is a sectorial operator, it is the infinitesimal generator of the analytic semigroup $E_{\Delta x}$. For further details see for example [104, 68].□

Theorem 3.7.3 *Let $E_{\Delta x}(t)$ be the linear semi-group generated by $-(1+i\nu)\widetilde{A}$ and let $V \in \mathbb{C}^J_{\text{per}}$. Then there exists C independent of Δx such that the discrete smoothing property holds, that is*

$$\|E_{\Delta x}(t)V\|_{H^\beta} \leq Ct^{-(\beta-\alpha)/2}\|V\|_{H^\alpha}, \ t>0, \ \ 0 \leq \alpha \leq \beta.$$

Proof A proof of this result may be inferred from the standard finite element results such as may be found in [17, 130] or [52] using the norm equivalence result (Theorem 3.7.1). In the context of finite differences see for example [19]. □

Gronwall Lemmas

The following lemmas are discrete versions of Gronwall lemmas we presented in Section 3.4.1.

Lemma 3.7.17 (Discrete Gronwall [45])

Let $0 \le y^n \le R$ for $0 \le n\Delta t \le T$. If for some constants $A_1, A_2, B, \alpha_1, \alpha_2, \beta > 0$ we have that

$$y^n \le A_1(n\Delta t)^{-1+\alpha_1} + A_2(n\Delta t)^{-1+\alpha_2} + B\Delta t \sum_{k=1}^{n} ((n-k+1)\Delta t)^{-1+\beta} y^k,$$

for $0 < n\Delta t \le T$, then there exists constants $\Delta t_0 = \Delta t_0(R, B, \beta)$ and $C = C(B, T, \alpha_1, \alpha_2, \beta)$ such that for $\Delta t < \Delta t_0$

$$y^n \le C\left(A_1(n\Delta t)^{-1+\alpha_1} + A_2(n\Delta t)^{-1+\alpha_2}\right), \quad 0 < n\Delta t \le T.$$

Proof We refer the reader to Lemma 7.1 of [45]. □

Lemma 3.7.18 (Discrete Uniform Gronwall [46])

Let Q^n, P^n, Y^n be three positive sequences satisfying:

$$\frac{Y^{n+1} - Y^n}{\Delta t} \le Q^n Y^{n+1} + P^n \quad \forall n > n_0$$

$$\sum_{n=k_0}^{N+k_0} \Delta t Q^n \le a_1(r), \quad \sum_{n=k_0}^{N+k_0} \Delta t P^n \le a_2(r), \quad \sum_{n=k_0}^{N+k_0} \Delta t Y^n \le a_3(r)$$

where

$$\sum_{n=k_0}^{N+k_0} \Delta t = r.$$

Then provided

$$\Delta t Q^n \le 1 - \delta \quad \forall n > n_0$$

we have

$$Y^{N+k_0+1} \le \left[\frac{a_3}{r} + a_2\right] \exp(a_1/\delta).$$

Proof See [46, Appendix 2].□

Lemma 3.7.19 (Composite Discrete Uniform Gronwall)

Suppose G^n, H^n, P^n, V^n, Δt^n are positive sequences satisfying $\forall n \geq n_0$ and $k_0 \geq n_0$

$$\frac{V^{n+1} - V^n}{\Delta t^n} \leq G^n V^n + H^n V^{n+1} + P^n$$

and

$$\sum_{n=k_0}^{N+k_0} \Delta t^n G^n \leq a_0(r), \ \sum_{n=k_0}^{N+k_0} \Delta t^n H^n \leq a_1(r), \ \sum_{n=k_0}^{N+k_0} \Delta t^n P^n \leq a_2(r)$$

$$\sum_{n=k_0}^{N+k_0} \Delta t^n V^n \leq a_3(r), \ \textit{where,} \ \sum_{n=k_0}^{N+k_0} \Delta t^n = r.$$

Then, provided $\Delta t^n H^n \leq 1 - \delta \ \forall n \geq n_0$

$$V^{N+k_0+1} \leq \left[\frac{a_3}{r} + a_2\right] \exp(a_0 + a_1/\delta) \ \forall k_0 \geq n_0. \tag{3.7.30}$$

Proof Rearranging the scheme we have,

$$V^{n+1}(1 - \Delta t^n H^n) \leq V^n(1 + \Delta t^n G^n) + \Delta t P^n.$$

Now let us define a sequence Q^n in the following manner:

$$Q^{n+1} = \frac{1 - \Delta t^n H^n}{1 + \Delta t^n G^n} Q^n$$

The conditions in the statement ensure that Q^n is a decreasing sequence. We use this new sequence and the fact that $(1 + \Delta t^n G^n) \geq 1$ to get that:

$$Q^{n+1} V^{n+1} \leq Q^n V^n + \frac{\Delta t^n P^n Q^n}{1 + \Delta t^n G^n} \leq Q^n V^n + \Delta t^n P^n Q^n,$$

which when we sum from $m (\geq n_0)$ to $N + k_0$ we get:

$$Q^{N+k_0+1} V^{N+k_0+1} - V^m Q^m \leq \sum_{n=m}^{N+k_0} \Delta t^n P^n Q^n.$$

Thus, using the definition of the Q^n's:

$$V^{N+k_0+1} \left\{ Q^m \prod_{n=m}^{N+k_0} \frac{1 - \Delta t^n H^n}{1 + \Delta t^n G^n} \right\} - Q^m V^m \leq Q^m \sum_{n=m}^{N+k_0} \Delta t^n P^n$$

and so,

$$V^{N+k_0+1} \leq \left(V^m + \sum_{n=m}^{N+k_0} \Delta t^n P^n \right) \prod_{n=m}^{N+k_0} \frac{1 + \Delta t^n G^n}{1 - \Delta t^n H^n}.$$

Now since

$$\Delta t^n H^n \leq 1-\delta,$$

$$(1-\Delta t^n H^n)^{-1} \leq \exp(\Delta t^n H^n/\delta), \text{ and } (1+\Delta t^n G^n) \leq \exp(\Delta t^n G^n),$$

we find

$$V^{N+k_0+1} \leq (V^m + a_2)\exp(\sum \Delta t^n (G^n + H^n/\delta)).$$

Hence, summing over m and dividing by r we find:

$$V^{N+k_0+1} \leq \left[\frac{a_3}{r} + a_2\right] \exp(a_0 + a_1/\delta). \qquad \square$$

Lipschitz Inequalities

The next lemma is a simple but useful algebraic relationship which allows us to get a Lipschitz inequality for the non–linear term (3.6.8).

Lemma 3.7.20 *For any $a, b \in \mathbb{C}$*

$$\left(|a|^2 a - |b|^2 b\right)\left(\overline{a} - \overline{b}\right) + \left(|a|^2\overline{a} - |b|^2\overline{b}\right)(a-b) \;\leq\; \left(|a|^2 + |b|^2\right)\left(|a-b|^2\right).$$

Proof This is just simple algebra:

$$\begin{aligned}
&\left(|a|^2 a - |b|^2 b\right)\left(\overline{a} - \overline{b}\right) + \left(|a|^2\overline{a} - |b|^2\overline{b}\right)(a-b) \\
&= |a|^4 - |a|^2 a\overline{b} - |b|^2 b\overline{a} + |b|^4 + |a|^4 - |a|^2 b\overline{a} - |b|^2 a\overline{b} + |b|^4 \\
&= |a|^2\left(|a|^2 - a\overline{b} - b\overline{a} + |b|^2\right) - |b|^2|a|^2 + |b|^2\left(|a|^2 - b\overline{a} - a\overline{b} + |b|^2\right) - |b|^2|a|^2 \\
&\leq \left(|a|^2 + |b|^2\right)\left(|a-b|^2\right). \qquad \square
\end{aligned}$$

Corollary 3.7.1

With $G(\cdot)$ defined in (3.6.8), for all $U, V \in L^2_{\Delta x}$, $\mu \in \mathbb{R}$, we have that

$$Re\left\{(1+i\mu) < G(|U|^2)U - G(|V|^2)V, U-V >\right\} \leq (1+\mu^2)^{1/2}(|U|^2_{L^\infty_{\Delta x}} + |V|^2_{L^\infty_{\Delta x}})|U-V|^2_{L^2_{\Delta x}}.$$

Proof Follows from Lemma 3.7.20 after writing the inner product as a summation and taking the real part. $\square$

Our next lemma shows that the non–linear term is Lipschitz continuous from $H^1_{\Delta x}$ to $L^2_{\Delta x}$.

Lemma 3.7.21 *With $G(\cdot)$ defined in* (3.6.8), *for all $U, V \in H^1_{\Delta x}$ we have that*

$$|G(|U|^2)U - G(|V|^2)V|_{L^2_{\Delta x}} \leq C(|U|_{L^\infty_{\Delta x}}, |V|_{L^\infty_{\Delta x}})|U - V|^2_{L^\infty_{\Delta x}}.$$

Proof We apply the inequality in $\mathbb{R}$ proved in Lemma 3.4.1

$$\begin{aligned}
|G(|U|^2)U - G(|V|^2)V|_{L^2_{\Delta x}} &= \sum_{j=0}^{J-1} \Delta x \left||U_j|^2 U_j - |V_j|^2 V_j\right|^2 \\
&\leq \frac{9}{4} \sum_{j=0}^{J-1} \Delta x \left(|U_j|^2 + |V_j|^2\right)^2 |U - V|^2 \\
&\leq \frac{9}{4} C(|U|_{L^\infty_{\Delta x}}, |V|_{L^\infty_{\Delta x}})|U - V|^2_{L^\infty_{\Delta x}}. \qquad \square
\end{aligned}$$

The following lemma will be used to prove a Lipschitz inequality in $H^1_{\Delta x}$ for the non–linear term.

Lemma 3.7.22 *For $U \in \mathbb{C}^J_{\text{per}}$ the following equality holds*

$$\begin{aligned}
&\delta_+(|U_j|^2 U_j) - \delta_+(|V_j|^2 V_j) \\
&= \left(U_{j+1}^2 + \overline{U}_{j+1} U_j + |U_j|^2\right) \delta_+(U_j - V_j) + \delta_+ V_j (U_{j+1} + V_{j+1} + \overline{V}_j)(U_{j+1} - V_{j+1}) \\
&\quad + \delta_+ V_j (U_{j+1} + U_j)(\overline{U}_j - \overline{V}_j) + \delta_+ V_j (U_j - V_j)\overline{V}_j.
\end{aligned}$$

Proof This is simply ugly algebra: first note that

$$\begin{aligned}
\delta_+(|U_j|^2 U_j) &= |U_{j+1}|^2 U_{j+1} - |U_j|^2 U_j \\
&= U_{j+1}^2 \overline{U}_{j+1} - U_j^2 \overline{U}_j \\
&= U_{j+1}^2 \delta_+ \overline{U}_j + \overline{U_j}(U_{j+1}^2 - U_j^2) \\
&= U_{j+1}^2 \delta_+ \overline{U}_j + \overline{U}_j \left\{(U_{j+1} - U_j)U_{j+1} + U_{j+1}U_j - U_j^2\right\} \\
&leq\ U_{j+1}^2 \delta_+ \overline{U_j} + \left(\overline{U_j} U_{j+1} + |U_j|^2\right) \delta_+ U_j.
\end{aligned}$$

We apply this to $\delta_+(|U_j|^2 U_j) - \delta_+(|V_j|^2 V_j)$ to get

$$\begin{aligned}
&\delta_+(|U_j|^2 U_j) - \delta_+(|V_j|^2 V_j) \\
&= U_{j+1}^2 \delta_+ \overline{U_j} + \left(\overline{U_j} U_{j+1} + |U_j|^2\right) \delta_+ U_j - \left\{V_{j+1}^2 \delta_+ \overline{V_j} + \left(\overline{V_j} V_{j+1} + |V_j|^2\right) \delta_+ V_j\right\} \\
&= U_{j+1}^2 \delta_+ \overline{U}_j - V_{j+1}^2 \delta_+ \overline{V_j} + \overline{U_j} U_{j+1} \delta_+ U_j - \overline{V_j} V_{j+1} \delta_+ V_j + |U_j|^2 \delta_+ U_j - |V_j|^2 \delta_+ V_j \\
&= U_{j+1}^2 \delta_+(\overline{U}_j - \overline{V}_j) + (U_{j+1}^2 - V_{j+1}^2)\delta_+ V_j + \overline{U}_j U_{j+1} \delta_+(U_j - V_j)
\end{aligned}$$

$$
\begin{aligned}
&+ (\overline{U}_j U_{j+1} - \overline{V}_j V_{j+1})\delta_+ V_j + |U_j|^2 \delta_+(U_j - V_j) + (|U_j|^2 - |V_j|^2)\delta_+ V_j \\
= \;& \left(U_{j+1}^2 + \overline{U}_{j+1} U_j + |U_j|^2\right) \delta_+(U_j - V_j) + \Big(U_{j+1}^2 - V_{j+1}^2 \\
&+ \overline{U}_j U_{j+1} - \overline{V}_j V_{j+1} + |U_j|^2 - |V_j|^2\Big) \delta_+ V_j \\
= \;& \left(U_{j+1}^2 + \overline{U}_{j+1} U_j + |U_j|^2\right) \delta_+(U_j - V_j) + \{(U_{j+1} - V_{j+1})(U_{j+1} + V_{j+1}) \\
&+ (\overline{U}_j - \overline{V}_j) U_{j+1} + (U_{j+1} - V_{j+1})\overline{V}_j + (\overline{U}_j - \overline{V}_j) U_j + \overline{V}_j (U_j - V_j)\Big\} \delta_+ V_j;
\end{aligned}
$$

and the lemma is proved.□

Corollary 3.7.2 *With $G(\cdot)$ defined in* (3.6.8), *for all $U, V \in H^1_{\Delta x}$ we have*

$$\|G(|U|^2)U - G(|V|^2)V\|_1 \leq C(\|U\|_1, \|V\|_1)\|U - V\|_1.$$

Proof We apply Lemma (3.7.22)

$$
\begin{aligned}
&\|G(|U|^2)U - G(|V|^2)V\|_1^2 = \sum_{j=0}^{J-1} \Delta x \left|\delta_+(|U_j|^2 U_j) - \delta_+(|V_j|^2 V_j)\right|^2 \\
\leq \;& 3 \sum_{j=0}^{J-1} \Delta x \left| \left(U_{j+1}^2 + \overline{U}_{j+1} U_j + |U_j|^2\right) \delta_+(U_j - V_j)\right|^2 \\
&+ 3 \sum_{j=0}^{J-1} \Delta x \left|\delta_+ V_j (U_{j+1} + U_j)(\overline{U}_j - \overline{V}_j)\right|^2 \\
&+ 3 \sum_{j=0}^{J-1} \Delta x \left|\delta_+ V_j (U_j - V_J)\overline{V}_j\right|^2 \\
\leq \;& 9|U|^2_{L^\infty_{\Delta x}} \|U - V\|_1^2 + 3|U - V|^2_{L^\infty_{\Delta x}} \left(|U|_{L^\infty} + 2|V|_{L^\infty}\right)^2 \|V\|_1^2 \\
&+ 12|U - V|^2_{L^\infty_{\Delta x}} |U|^2_{L^\infty_{\Delta x}} \|V\|_1^2 + 3|U - V|^2_{L^\infty_{\Delta x}} |V|^2_{L^\infty_{\Delta x}} \|V\|_1^2 \\
\leq \;& 9|U|^2_{L^\infty_{\Delta x}} \|U - V\|_1^2 + \|V\|_1^2 \left\{15|U|^2_{L^\infty_{\Delta x}} + 13|U|_{L^\infty_{\Delta x}} |V|_{L^\infty_{\Delta x}} + 15|V|^2_{L^\infty_{\Delta x}}\right\} |U - V|^2_{L^\infty_{\Delta x}}.
\end{aligned}
$$

Thus we find by Lemma 3.7.8 that

$$\|G(|U|^2)U - G(|V|^2)V\|_1^2 \leq C(U, V)\|U - V\|_1^2,$$

and the lemma is proved.□

Chapter 4

Finite–Dimensional Dynamics

In this chapter we consider the semi–discrete problem **SD** (3.6.9) and the fully discrete schemes **DI** (3.6.12) and **DEI** (3.6.16) assuming we are given some initial condition $U^0 \in L^2_{\Delta x}$. It was shown in section 3.4.2 that the continuous complex Ginzburg–Landau equation has a global attractor and that solutions $U(t)$ are of Gevrey regularity τ for t sufficiently large. We shall prove below that each of these discretizations has a global attractor. For the semi–discrete case we prove that the solutions are in a discrete Gevrey class. This will allow us to prove upper–semicontinuity in the $H^1_{\Delta x}$ norm of the approximate attractors to the continuous global attractor as $\Delta x \to 0$. Upper–semicontinuity is a particular form of convergence result. Roughly speaking we prove that as our spatial and temporal mesh sizes tend to zero the discrete attractor is arbitrarily close to the continuous attractor. In the limit at "$\Delta x = 0$" the approximate attractor is contained in the true attractor. General results on upper–semicontinuity may be found for example in [64] for finite element and spectral approximations or for linear multi–step methods for convection–diffusion equations in [73].

Yin Yan [137] looks at upper–semicontinuity for finite difference spatial approximations for the 2 dimensional Navier Stokes Equations and is able to prove upper–semicontinuity in L^2 by interpolating the nodal values by piecewise linear functions. By using the Gevrey property we are able to obtain upper–semicontinuity in $H^1_{\Delta x}$ for initial data in $L^2_{\Delta x}$.

The chapter concludes with a section of numerical results and observations for the schemes **DFE**, (3.6.14), **DE**, (3.6.15), **DEI**, (3.6.16), **DI**, (3.6.12).

Notation

▷ Let X be a Banach space with norm $|\bullet|_X$. We define the ϵ–neighbourhood of $A \subset X$ denoted $N(A, \epsilon)$ by

$$N(A, \epsilon) = \{v \in X : \mathrm{dist}_X(v, A) < \epsilon\},$$

where we recall that dist_X denotes the semi–distance in X defined by

$$\mathrm{dist}_X(u, A) := \inf_{v \in A} \|u - v\|_X.$$

▷ We define the ball centre 0, radius ρ in $L^2_{\Delta x} (= H^0_{\Delta x})$ by

$$\mathcal{B}_0(\rho) := \left\{ V \in \mathbb{C}^J_{\mathrm{per}} \;:\; |V|_{L^2_{\Delta x}} \le \rho \right\};$$

the ball in $H^1_{\Delta x}$ centre 0 radius ρ by

$$\mathcal{B}_1(\rho) := \left\{ V \in \mathbb{C}^J_{\mathrm{per}} \;:\; \|V\|_{H^1_{\Delta x}} \le \rho \right\};$$

and finally the ball in $G_{\tau, \Delta x}$ centre 0, radius ρ by

$$\mathcal{B}_\tau(\rho) := \left\{ V \in \mathbb{C}^J_{\mathrm{per}} \;:\; |\widetilde{A}^{1/2} e^{\tau \widetilde{A}^{1/2}} V|_{L^2_{\Delta x}} \le \rho \right\}.$$

4.1 The Semi-Discrete Problem

The immediate aim is to prove that the set of ordinary differential equations (3.6.9) which arise from the spatial discretisation of (3.1.1), satisfies semi–discrete versions of **C1 - C4** established in section 3.4. These semi–discrete versions will be denoted **SD1 - SD4**; loosely then

SD1 The family of solution operators forms a continuous semi–group in $L^2_{\Delta x}$

SD2 There is an absorbing set $\mathcal{B}_0(\rho_0)$, with ρ_0 independent of Δx.

SD3 There is an absorbing set $\mathcal{B}_1(\rho_1)$, with ρ_1 independent of Δx.

SD4 Existence of a global attractor.

These will be made precise in the statement of the Theorems. Recall that we reduced the infinite–dimensional problem to a finite–dimensional problem. The ordinary differential equations which result from that reduction are set in $\mathbb{C}^J_{\mathrm{per}}$ or equivalently in

$\mathbb{R}^{2J}$. The finite dimensionality means that all norms are equivalent. Hence, once we have proved **SD2**, we might be tempted to appeal to this norm equivalence and to use an inverse inequality to bound the $H^1_{\Delta x}$ norm to achieve **SD3**. This would lead to a bound on the $H^1_{\Delta x}$ norm of the form

$$\|U(t)\|_{H^1_{\Delta x}} \leq \Delta x^{-1} C \quad t > t_0$$

where C is a constant. This form of bound is unsatisfactory. Certainly it does give the existence of an absorbing set in $H^1_{\Delta x}$ for any *fixed* Δx, but as $\Delta x \to 0$ the bound fails to hold. As we wish to study the convergence of our discrete global attractor to the original we seek estimates which will hold uniformly as $\Delta x \to 0$.

The second requirement we seek to fulfill is that our estimates for the absorbing sets are *not* of the form

$$\|U(t)\| \leq C(U^0).$$

This form is undesirable because we already know that for the continuous equation the absorbing sets are independent of initial data and thus the global attractor for (3.1.1) attracts *all* initial conditions which are in $L^2_{\Delta x}$. Therefore we aim to find discrete estimates which are also independent of the initial data.

Our first Lemma forms the backbone of the proof of both **SD1** and **SD2**.

Lemma 4.1.1 *If $U(t)$ is a solution of* (3.6.9) *for $t \geq 0$ with initial condition $U^0 \in L^2_{\Delta x}$ then the $L^2_{\Delta x}$ norm of $U(t)$ satisfies*

$$\frac{1}{2}\frac{d}{dt}|U|^2_{L^2_{\Delta x}} = R|U|^2_{L^2_{\Delta x}} - \|U\|_1^2 - |U|^4_{L^4_{\Delta x}}; \tag{4.1.1}$$

and hence

$$|U(t)|^2_{L^2_{\Delta x}} \leq |U(0)|^2_{L^2_{\Delta x}} + 2|R| \quad \forall t \geq 0. \tag{4.1.2}$$

Proof Take the $L^2_{\Delta x}$ inner product of (3.6.9) with U to get :

$$\left\langle \frac{dU}{dt}, U \right\rangle = R < U, U > -(1+i\nu) < M^{-1}AU, U > -(1+i\mu) < G(|U|^2)U, U >$$

and take the real part. The diffusion term is dealt with by (3.7.15), the time derivative and the linear term are straightforward. For the non–linear term note that

$$\langle G(|U|^2)U, U\rangle = \sum_{j=0}^{J-1} \Delta x |U_j|^2 U_j \overline{U_j} = \sum_{j=0}^{J-1} \Delta x |U_j|^4 = |U|^4_{L^4_{\Delta x}};$$

which yields equation (4.1.1).

In order to get (4.1.2) we have three possible cases to consider

1. If $R < 0$ then
$$\frac{d}{dt}|U|^2_{L^2_{\Delta x}} \leq 2R|U|^2_{L^2_{\Delta x}}.$$
This we can solve to find
$$|U(t)|^2_{L^2_{\Delta x}} \leq |U(0)|^2_{L^2_{\Delta x}} \exp(2Rt). \tag{4.1.3}$$
Since $R < 0$ the result is trivially true.

2. If $R = 0$ then Lemma 3.7.7 applied to (4.1.1) yields:
$$\frac{1}{2}\frac{d}{dt}|U|^2_{L^2_{\Delta x}} \leq -|U|^4_{L^2_{\Delta x}}.$$
Hence
$$\frac{1}{|U(0)|^2_{L^2_{\Delta x}}} + t \leq \frac{1}{|U(t)|^2_{L^2_{\Delta x}}}$$
which is equivalent to
$$|U(t)|^2_{L^2_{\Delta x}} \leq \frac{|U(0)|^2_{L^2_{\Delta x}}}{1 + t|U(0)|^2_{L^2_{\Delta x}}}. \tag{4.1.4}$$
Since $t \geq 0$ we get:
$$|U(t)|^2_{L^2_{\Delta x}} \leq |U(0)|^2_{L^2_{\Delta x}}$$
and (4.1.2) is satisfied.

3. If $R > 0$ then we use that $\forall s \in \mathbb{R}$
$$\frac{1}{2}s^4 - 2Rs^2 \geq -2R^2 \tag{4.1.5}$$
to bound the non–linear term in equation (4.1.1). This yields that
$$\begin{aligned}\frac{d}{dt}|U|^2_{L^2_{\Delta x}} &\leq 2R|U|^2_{L^2_{\Delta x}} - 2\|U\|^2_1 - |U|^4_{L^4_{\Delta x}} - 4R|U|^2_{L^2_{\Delta x}} + 4R^2 \\ &= -2R|U|^2_{L^2_{\Delta x}} - 2\|U\|^2_1 - |U|^4_{L^4_{\Delta x}} + 4R^2, \end{aligned} \tag{4.1.6}$$
and so
$$\frac{d}{dt}|U|^2_{L^2_{\Delta x}} \leq -2R|U|^2_{L^2_{\Delta x}} + 4R^2.$$
Now apply the standard Gronwall inequality (see Lemma 3.4.5) to get for all $t > 0$
$$|U(t)|^2_{L^2_{\Delta x}} \leq |U(0)|^2_{L^2_{\Delta x}} \exp(-2Rt) + 2R(1 - \exp(-2Rt)). \tag{4.1.7}$$

Hence the Lemma is proved. □

We now prove **SD1**.

Theorem 4.1.1 (SD1) *For each $U^0 \in L^2_{\Delta x}$ there exists a unique solution $U \in C^1(0,T;L^2_{\Delta x})$ of (3.6.9) for all $T > 0$. The mapping $U^0 \to U(t)$ is continuous in $L^2_{\Delta x}$ for each $t \in (0,T)$ and hence there exists a family of solution operators*

$$\{S_{\Delta x}(t)\}_{t\geq 0}$$

defined by $S_{\Delta x}(t)U_0 \equiv U(t)$which forms a continuous semi-group on $L^2_{\Delta x}$.

Proof

Consider (3.6.9) as a set of ordinary differential equations in $\mathbb{R}^{2J}$. Then the right hand side of (3.6.9) is locally Lipschitz with constant K_1, and local existence and uniqueness is immediate from the standard theory such as in [67] .

To prove existence for any $T > 0$ we appeal to the inverse inequality -

$$\forall\, V(\cdot) \,\in\, L^2_{\Delta x} \quad |V(\cdot)|_{L^\infty_{\Delta x}} \leq \Delta x^{-1/2}|V(\cdot)|_{L^2_{\Delta x}}, \tag{4.1.8}$$

proved in Lemma 3.7.12.

If we apply this to $U(t)$ and use Lemma 4.1.1 to bound the $L^2_{\Delta x}$ norm we get

$$\Delta x|U(t)|^2_{L^\infty_{\Delta x}} \leq |U(0)|^2_{L^\infty_{\Delta x}} + 2|R|.$$

Thus we have an a priori bound on the Lipschitz constant K_1 for all $t > 0$ and global existence follows.

In order to prove continuity with respect to initial data consider $U(t) - V(t)$ where $V(t)$ also satisfies (3.7.2) but with initial data V_0. Taking the inner product with $U - V$ and the real part, dealing with the non–linear term by Corollary 3.7.1 and the other terms as we did in proving (4.1.1) we get

$$\frac{1}{2}\frac{d}{dt}|U-V|^2_{L^2_{\Delta x}} \leq R|U-V|^2_{L^2_{\Delta x}} - \|U-V\|^2_1 + \left(|U|^2_{L^\infty_{\Delta x}} + |V|^2_{L^\infty_{\Delta x}}\right)|U-V|^2_{L^2_{\Delta x}}.$$

Thus

$$\frac{1}{2}\frac{d}{dt}|U-V|^2_{L^2_{\Delta x}} \leq C(\,|U(0)|_{L^\infty_{\Delta x}}, |V(0)|_{L^\infty_{\Delta x}}, R, \mu)\;|U-V|^2_{L^2_{\Delta x}}$$

where $C(\,|U(0)|_{L^\infty_{\Delta x}}, |V(0)|_{L^\infty_{\Delta x}}, R, \mu)$ is constant.

Thus by Lemma 3.7.1 and the standard Gronwall lemma 3.4.5 we obtain

$$|U(t) - V(t)|^2_{L^2_{\Delta x}} \leq C(\,T, |U(0)|_{L^\infty_{\Delta x}}, |V(0)|_{L^\infty_{\Delta x}}, R, \mu)|U(0) - V(0)|^2_{L^2_{\Delta x}} \quad \forall t \in (0,T).$$

□

The next theorem establishes **SD2**.

Theorem 4.1.2 (SD2) *There exists a constant $\rho_0 = \rho_0(R) > 0$, independent of Δx, such that the ball $\mathcal{B}_0(\rho_0)$ is absorbing and positively invariant for the semi-group $\{S_{\Delta x}(t)\}_{t\geq 0}$. That is for any bounded set $B \subset \mathcal{B}_0(\rho)$, $\exists \rho_0 > 0$ and $t_0 = t_0(\rho, \rho_0)$:*

$$S_{\Delta x}(t)B \subset \mathcal{B}_0(\rho_0) \quad \forall t > t_0,$$

and ρ_0 is independent of Δx.

Proof Recall (4.1.1) and consider the three possible cases for R.

1. For $R < 0$ Lemma 4.1.1 equation (4.1.3) immediately yields trivial dynamics. Thus for any $\epsilon > 0$ we have $\rho_0 = \epsilon$, and

$$t_0 = \begin{cases} \frac{1}{2R}\log\left(\frac{\rho_0}{\rho}\right) & \rho_0 > \rho \\ 0 & \rho_0 < \rho \end{cases}.$$

2. For $R = 0$ equation (4.1.4) immediately gives algebraic decay to zero. For any $\epsilon > 0$ we have $\rho_0 = \epsilon$ for any $\epsilon > 0$ and

$$t_0 = \begin{cases} \frac{1}{\rho_0^2} - \frac{1}{\rho^2} & \rho_0 > \rho \\ 0 & \rho_0 \leq \rho \end{cases}.$$

3. For $R > 0$ note that (4.1.7) gives

$$\limsup_{t\to\infty} |U(t)|^2_{L^2_{\Delta x}} \leq \rho'^2$$

where $\rho'^2 := 2R$.

Therefore the ball of $L^2_{\Delta x}$, $\mathcal{B}_0(\rho_0)$, $\rho_0 > \rho'$, is positively invariant and is absorbing for the semi-group $S_{\Delta x}(t)$. If $B \subset \mathcal{B}_0(\rho)$ then

i) for $\rho < \rho_0, S_{\Delta x}(t)B \subset \mathcal{B}_1 \forall\, t \geq t_0 = 0$

ii) for $\rho \geq \rho_0, S_{\Delta x}(t)B \subset \mathcal{B}_1$ for $t \geq t_0 = t_0(\mathcal{B}_1, B)$, where

$$t_0 = \frac{1}{2R} \log \left\{ \frac{\rho^2}{\rho_0^2 - \rho'^2} \right\}.$$

Hence the theorem proved. □

Note

We can integrate (4.1.6) between t and $t + r$ for $t > t_0$, to get:

$$\int_t^{t+r} \frac{d}{dt} |U(t)|^2_{L^2_{\Delta x}} dt \leq - \int_t^{t+r} (2R|U|^2_{L^2_{\Delta x}} + |U|^4_{L^4_{\Delta x}} + 2\|U\|_1^2 - 4R^2) dt;$$

which using the uniform bound on the $L^2_{\Delta x}$ norm from Theorem 4.1.2 becomes

$$\int_t^{t+r} \left\{ 2R|U|^2_{L^2_{\Delta x}} + 2\|U\|_1^2 + |U|^4_{L^4_{\Delta x}} \right\} \leq \rho_0^2 + 4R^2 r \quad , t > t_0. \tag{4.1.9}$$

This will be of use when applying the uniform Gronwall lemma to prove the existence of an absorbing set in $H^1_{\Delta x}$.

We recall that since we are in finite dimensions, the existence of the $L^2_{\Delta x}$ absorbing set immediately yields the existence of a global attractor and, by inverse inequalities, the existence of absorbing sets in $H^1_{\Delta x}$. However, as was remarked before, we wish to prove convergence of the global attractors as $\Delta x \to 0$ and so seek bounds for $H^1_{\Delta x}$ which hold uniformly as $\Delta x \to 0$.

For the remainder of this section R is assumed to be strictly positive since for $R \leq 0$ both the continuous equation and the semi-discrete system have trivial dynamics.

Theorem 4.1.3 (SD3) *There exists a constant $\rho_1 = \rho_1(R) > 0$, independent of Δx, such that the ball $\mathcal{B}_1(\rho_1)$ is absorbing and positively invariant for the semi-group $\{S_{\Delta x}(t)\}_{t \geq 0}$. That is, for $B \subset \mathcal{B}_0(\rho)$, $\exists \rho_1 > 0$ and $t_1 = t_1(\rho, \rho_1)$:*

$$S_{\Delta x}(t)B \subset \mathcal{B}_1(\rho_1) \;\; \forall t > t_1,$$

where ρ_1 is independent of Δx.

Proof The aim now is to show that we have an absorbing set in the discrete $H^1_{\Delta x}$ norm, so we seek a bound on the $H^1_{\Delta x}$ semi–norm $\|\cdot\|_1$ independent of the spatial mesh size Δx. To do this take the discrete Dirichlet inner product (defined in (3.7.14)) of (3.6.9) with U and take the real part:

$$\frac{1}{2}\frac{d}{dt}\|U\|_1^2 = R\|U\|_1^2 - |M^{-1}AU|_{L^2_{\Delta x}}^2 + \operatorname{Re}\left\{(1+i\mu)\sum_{j=0}^{J-1} \Delta x \, |U_j|^2 U_j \delta^2 \overline{U}_j^{n+1}\right\}.$$

Now use summation by parts and the boundary conditions to obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|U\|_1^2 &= R\|U\|_1^2 - |M^{-1}AU|_{L^2_{\Delta x}}^2 \\
&\quad - \operatorname{Re}\left\{(1+i\mu)\sum_{j=0}^{J-1} \Delta x \, (U_{j+1}|U_{j+1}|^2 - U_j|U_j|^2)\frac{(\overline{U_{j+1}} - \overline{U_j})}{\Delta x^2}\right\} \\
&= R\|U\|_1^2 - |M^{-1}AU|_{L^2_{\Delta x}}^2 \\
&\quad - \operatorname{Re}\left\{(1+i\mu)\sum_{j=0}^{J-1} \Delta x \left(\frac{(U_{j+1} - U_j)}{\Delta x}(|U_{j+1}|^2 + |U_j|^2)\frac{(\overline{U_{j+1}} - \overline{U_j})}{\Delta x}\right.\right. \\
&\qquad \left.\left. + (|U_{j+1}|^2 U_j - U_{j+1}|U_j|^2)(\frac{\overline{U}_{j+1} - \overline{U}_j}{\Delta x^2}) \right)\right\} \\
&\le R\|U\|_1^2 - |M^{-1}AU|_{L^2_{\Delta x}}^2 \\
&\quad - \operatorname{Re}\left\{(1+i\mu)\sum_{j=0}^{J-1} \Delta x \left(\left|\frac{U_{j+1} - U_j}{\Delta x}\right|^2 (|U_{j+1}|^2 + |U_j|^2)\right.\right. \\
&\qquad \left.\left. + U_{j+1}U_j\frac{(\overline{U}_{j+1} - \overline{U}_j)^2}{\Delta x^2}\right)\right\}. \qquad (4.1.10)
\end{aligned}$$

Completing the square on the last term gives

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|U\|_1^2 \\
&\le R\|U\|_1^2 - |M^{-1}AU|_{L^2_{\Delta x}}^2 + (1+\mu^2)^{1/2}\sum_{j=0}^{J-1} \Delta x \left|\frac{U_{j+1} - U_j}{\Delta x}\right|^2 (|U_{j+1}|^2 + |U_j|^2) \\
&\quad +(1+\mu^2)^{1/2}\sum_{j=0}^{J-1} \Delta x \frac{(|U_{j+1}|^2 + |U_j|^2)}{2}\left|\frac{\overline{U_{j+1}} - \overline{U_j}}{\Delta x}\right|^2 \\
&\le R\|U\|_1^2 - |M^{-1}AU|_{L^2_{\Delta x}}^2 + \frac{3}{2}(1+\mu^2)^{1/2}\sum_{j=0}^{J-1} \Delta x \left|\frac{U_{j+1} - U_j}{\Delta x}\right|^2 (|U_j|^2 + |U_{j+1}|^2).
\end{aligned}$$

Now apply Schwarz's inequality on the last term and use the periodicity to get

$$\frac{1}{2}\frac{d}{dt}\|U\|_1^2 \le R\|U\|_1^2 - |M^{-1}AU|_{L^2_{\Delta x}}^2 + 3(1+\mu^2)^{1/2}|U|_{L^4_{\Delta x}}^2 |DU|_{L^4_{\Delta x}}^2. \quad (4.1.11)$$

At this point we call upon Lemma 3.7.11 which bounds the $L^4_{\Delta x}$ norm of DU so that (4.1.11) becomes

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|U\|_1^2 &\leq R\|U\|_1^2 - |M^{-1}AU|^2_{L^2_{\Delta x}} \\ &\quad + 3\sqrt{6}(1+\mu^2)^{1/2}|U|^2_{L^4_{\Delta x}}\|U\|_1\left\{|U|^2_{L^2_{\Delta x}} + |M^{-1}AU|^2_{L^2_{\Delta x}}\right\}^{1/2} \quad (4.1.12)\end{aligned}$$

Complete the square on the last term to find that

$$\frac{1}{2}\frac{d}{dt}\|U\|_1^2 \leq R\|U\|_1^2 - |M^{-1}AU|^2_{L^2_{\Delta x}} + \frac{54}{2}(1+\mu^2)|U|^4_{L^4_{\Delta x}}\|U\|_1^2 + \frac{1}{2}\left\{|U|^2_{L^2_{\Delta x}} + |M^{-1}AU|^2_{L^2_{\Delta x}}\right\};$$

and hence

$$\frac{d}{dt}\|U\|_1^2 \leq (2R + 54(1+\mu^2)|U|^4_{L^4_{\Delta x}})\|U\|_1^2 + |U|^2_{L^2_{\Delta x}}. \qquad (4.1.13)$$

All that remains is to apply the uniform Gronwall Lemma 3.4.6 with

$$y = \|U\|_1^2, \quad g = 2R + 54(1+\mu^2)|U|^4_{L^4_{\Delta x}}, \quad h = |U|^2_{L^2_{\Delta x}};$$

and use the integral bound (4.1.9) to find the constants a_1, a_2 and a_3

$$\begin{aligned} a_1 &= 2Rr + 54(1+\mu^2)\left\{\rho_0^2 + 4R^2 r\right\}, \\ a_2 &= \rho_0^2, \\ a_3 &= \frac{1}{2}(\rho_0^2 + 4R^2 r). \end{aligned}$$

Therefore

$$\|U(t)\|_1^2 \leq \left(\frac{a_3}{r} + a_2\right)\exp(a_1) \ \forall t \geq t_0 + r; \qquad (4.1.14)$$

where r is an arbitrary positive number and t_0 is as in Theorem 4.1.2. Thus we obtain absorbing balls in $H^1_{\Delta x}$, with $t_1 > t_0 + r$ and ρ_1 such that

$$\rho_1^2 > \rho_0^2 + \left(\frac{a_3}{r} + a_2\right)\exp(a_1). \quad \square$$

We now state the theorem on the existence of a global attractor for (3.6.9).

Theorem 4.1.4 (SD4) *The dynamical system determined by the semi-discrete complex Ginzburg-Landau equation possesses a global attractor $\mathcal{A}_{\Delta x}$ given by*

$$\mathcal{A}_{\Delta x} = \omega(\mathcal{B}_1(\rho_1))$$

and hence is bounded in $H^1_{\Delta x}$, independently of Δx.

Proof

All that is required is to show the conditions for Theorem 1.2.1 hold. Theorems 4.1.2 and 4.1.3 give us all we require. The only comment to make is that if our initial data in B is only bounded in $L^2_{\Delta x}$ the analysis of Theorem 4.1.3 still holds and $S_{\Delta x}(t)B \subset B_1$ for $t \geq t_0 + r$. □

Notes

- Now that we have uniform estimates for the $L^2_{\Delta x}$ and $H^1_{\Delta x}$ norms of solutions $U(t)$ for $t > t_1$, we can apply Lemma 3.7.8 to obtain $L^\infty_{\Delta x}$ bounds on $U(t)$ inside the $H^1_{\Delta x}$ absorbing set, (i.e. for all $t > t_1$).
- We have established in Theorems 4.1.2–4.1.4 the existence of $L^2_{\Delta x}$ absorbing balls, $H^1_{\Delta x}$ absorbing balls and a global attractor without imposing a restriction on the spatial mesh size Δx.

4.1.1 Alternative method to find $H^1_{\Delta x}$ absorbing sets

Here we adapt the method used by Doering et al [41] to prove the existence of an absorbing set in the continuous space H^1 to prove the existence of an absorbing set in the discrete space $H^1_{\Delta x}$. This analysis treats the non–linear term more carefully so that improved estimates are found for particular ranges of parameter values.

Theorem 4.1.5 *Suppose we have a solution $U(t)$ of (3.6.9), and we have an $L^2_{\Delta x}$ absorbing set, so that t_0 and ρ_0 are as in Theorem 4.1.2. Then the following three inequalities hold for $t > t_0 + r$, any $r > 0$*

1. *For $|\mu|^2 \leq \sqrt{3}$ we have*

$$\|U(t)\|_1^2 \leq \|U(t_0)\|_1^2 e^{-(t-t_0)2R} + R\rho_0^2\left(1 - e^{-R(t-t_0)}\right)$$

or

$$\|U(t)\|_1^2 \leq \left(\frac{a_3'}{r} + a_2'\right)\exp(a_1')$$

with

$$a_1' = -2Rr, a_2' = 2R^2\rho_0^2 r, a_3 = \frac{1}{2}\left(\rho_0^2 + 4R^2 r\right).$$

2. For $|\mu|^2 > \sqrt{3}$ $\exists$ a constant $C = C(R, \mu, \rho_0)$ such that $\forall t > t_0$,

$$\|U(t)\|_1^2 \leq (\|U(t_0)\|_1 - 2C\rho_0^2)\exp\{-C(t-t_0)\} + 2C\rho_0^2.$$

Or, for $t \geq t_0 + r$ we have

$$\|U(t)\|_1^2 \leq \left(\frac{a_3''}{r} + a_2''\right)\exp(a_1'')$$

with constants

$$a_1'' = Cr, \quad a_2'' = C, \quad a_3'' = \frac{1}{2}\left(\rho_0^2 + 4R^2 r\right).$$

3. For purely real diffusion (i.e. $\nu = 0$) we have that ,

$$\lim_{t\to\infty} |U|_{L^\infty_{\Delta x}} < R.$$

Proof From equation (4.1.10) we have

$$\begin{aligned}\frac{1}{2}\frac{d\|U\|_1^2}{dt} &= R\|U\|_1^2 - |M^{-1}AU|_{L^2_{\Delta x}}^2 \\ &- \operatorname{Re}\left\{(1+i\mu)\sum_{j=0}^{J-1}\Delta x[\,|\frac{U_{j+1}-U_j}{\Delta x}|^2(|U_{j+1}|^2+|U_j|^2) + U_{j+1}U_j(\frac{\overline{U_{j+1}}-\overline{U_j}}{\Delta x})^2\,]\right\}.\end{aligned}$$

Using Lemma 3.7.10 to bound $-|M^{-1}AU|_{L^2_{\Delta x}}$ above and expand the non–linear term

$$\begin{aligned}\frac{1}{2}\frac{d\|U\|_1^2}{dt} &\leq R\|U\|_1^2 - \frac{\|U\|_1^4}{|U|_{L^2_{\Delta x}}^2} - \sum_{j=0}^{J-1}\Delta x|\frac{U_{j+1}-U_j}{\Delta x}|^2|U_{j+1}|^2 - \sum_{j=0}^{J-1}\Delta x|\frac{U_{j+1}-U_j}{\Delta x}|^2|U_j|^2 \\ &\quad + |1+i\mu|\sum_{j=0}^{J-1}\Delta x|U_{j+1}U_j|\left|\frac{U_{j+1}-U_j}{\Delta x}\right|^2.\end{aligned}$$

Now complete the square and use the boundary conditions on the final term

$$\begin{aligned}\frac{1}{2}\frac{d\|U\|_1^2}{dt} &\leq R\|U\|_1^2 - \frac{\|U\|_1^4}{|U|_{L^2_{\Delta x}}^2} - \sum_{j=0}^{J-1}\Delta x\left|\frac{U_{j+1}-U_j}{\Delta x}\right|^2|U_{j+1}|^2 - \sum_{j=0}^{J-1}\Delta x\left|\frac{U_{j+1}-U_j}{\Delta x}\right|^2|U_j|^2 \\ &\quad + \frac{(1+\mu^2)^{1/2}}{2}\sum_{j=0}^{J-1}\Delta x(|U_{j+1}|^2+|U_j|^2)\left|\frac{U_{j+1}-U_j}{\Delta x}\right|^2. \end{aligned} \qquad (4.1.15)$$

There are now three possible cases to consider.

Case 1 If $|\mu| \leq \sqrt{3}$ then

$$-1 + \frac{(1+\mu^2)^{1/2}}{2} \leq 0$$

and we can discard the terms arising from the non–linearity completely so (4.1.15) reduces to

$$\frac{1}{2}\frac{d\|U\|_1^2}{dt} \leq R\|U\|_1^2 - \frac{\|U\|_1^4}{|U|^2_{L^2_{\Delta x}}}.$$

Now for $t > t_0$ from the $L^2_{\Delta x}$ absorbing set property Theorem 4.1.2 $|U|^2_{L^2_{\Delta x}} \leq \rho_0^2$. Hence for $t > t_0$

$$\frac{1}{2}\frac{d}{dt}\|U\|_1^2 \;\leq\; R\|U\|_1^2 - \frac{\|U\|_1^4}{\rho_0^2}. \tag{4.1.16}$$

Noting that

$$\left(\frac{\|U\|_1^2}{\rho_0} - R\rho_0\right)^2 = \frac{\|U\|_1^4}{\rho_0^2} - 2R\|U\|_!^2 + R^2\rho_0^2 \geq 0,$$

we find that

$$\frac{1}{2}\frac{d}{dt}\|U\|_1^4 \;\leq\; R^2\rho_0^2 - R\|U\|_1^2. \tag{4.1.17}$$

We can now apply the Gronwall lemma 3.4.5 with

$$y = \|U\|_1^2,\;\; g = -2R,\;\; h = 2R^2\rho_0^2$$

to get for $t > t_0$

$$\|U(t)\|_1^2 \leq \|U(t_0)\|_1^2 e^{-(t-t_0)2R} + R\rho_0^2\left(1 - e^{-R(t-t_0)}\right).$$

Thus, if $\|U(t_0)\|_1 \leq R\rho_0^2$, we have

$$\|U(t)\|_1^2 \leq R\rho_0 \qquad \forall\, t > t_0.$$

Alternatively, we can apply the uniform Gronwall lemma 3.4.6 with y, g, h as above and the constants $a_1' = -2Rr$,$a_2' = 2R^2\rho_0^2 r$ and $a_3' = \frac{1}{2}\left(\rho_0^2 + 4R^2 r\right)$ found from (4.1.9). This yields for $t \geq t_0 + r$,

$$\|U(t)\|_1^2 \leq \left(\frac{a_3'}{r} + a_2'\right)\exp(a_1'). \tag{4.1.18}$$

This inequality requires no assumption on the semi–norm to get a bound for all $t \geq t_0 + r$.

Case 2 If $|\mu| > \sqrt{3}$ then we let

$$\delta = \max\left\{0, -2 + (1+\mu^2)^{1/2}\right\}.$$

This allows the terms from the non-linearity to be controlled in the following manner

$$\begin{aligned}
-\sum_{j=0}^{J-1} \Delta x \left|\frac{U_{j+1}-U_j}{\Delta x}\right|^2 |U_{j+1}|^2 + \frac{(1+\mu^2)^{1/2}}{2} \sum_{j=0}^{J-1} \Delta x |U_{j+1}|^2 \left|\frac{U_{j+1}-U_j}{\Delta x}\right|^2 \\
\leq \frac{1}{2}\delta \sum_{j=0}^{J-1} \Delta x \left|\frac{U_{j+1}-U_j}{\Delta x}\right|^2 |U_{j+1}|^2 \\
\leq \frac{1}{2}\delta |U|^2_{L^\infty_{\Delta x}} \|U\|_1^2.
\end{aligned}$$

Now apply Lemma 3.7.8 to bound the $L^\infty_{\Delta x}$ norm

$$\begin{aligned}
-\sum_{j=0}^{J-1} \Delta x \left|\frac{U_{j+1}-U_j}{\Delta x}\right|^2 |U_{j+1}|^2 + \frac{(1+\mu)^{1/2}}{2} \sum_{j=0}^{J-1} \Delta x |U_{j+1}|^2 \left|\frac{U_{j+1}-U_j}{\Delta x}\right|^2 \\
\leq \frac{1}{2}\delta \left\{|U|^2_{L^2_{\Delta x}} + \sqrt{2}|U|_{L^2_{\Delta x}}\|U\|_1\right\} \|U\|_1^2.
\end{aligned}$$

Thus (4.1.15) becomes

$$\begin{aligned}
\frac{1}{2}\frac{d\|U\|_1^2}{dt} &\leq R\|U\|_1^2 - \frac{\|U\|_1^4}{|U|^2_{L^2_{\Delta x}}} + \delta|U|^2_{L^2_{\Delta x}}\|U\|_1^2 + \delta\sqrt{2}|U|_{L^2_{\Delta x}}\|U\|_1^3 \\
&= (R + \delta|U|^2_{L^2_{\Delta x}})\|U\|_1^2 - \frac{\|U\|_1^4}{|U|^2_{L^2_{\Delta x}}} + \delta\sqrt{2}|U|_{L^2_{\Delta x}}\|U\|_1^3.
\end{aligned}$$

Now complete the square to get for $t > t_0$

$$\begin{aligned}
\frac{1}{2}\frac{d\|U\|_1^2}{dt} &\leq (R + \delta\rho_0^2 + 2\delta^2\rho_0^4)^2 2\rho_0^2 - (R + \delta\rho_0^2 + 2\delta^2\rho_0^4)\|U\|_1^2 \\
&= C(R,\mu,\rho_0)^2 2\rho_0^2 - C(R,\mu,\rho_0)\|U\|_1^2,
\end{aligned}$$

to which we can apply either the standard or uniform Gronwall inequality.

If we apply the standard Gronwall lemma 3.4.5 we find

$$\|U(t)\|_1^2 \leq (\|U(t_0)\|_1 - 2C\rho_0^2)\exp\{-C(t-t_0)\} + 2C\rho_0^2$$

for all $t \geq t_0$.

Alternatively we can apply the uniform Gronwall lemma 3.4.6 with $a_1'' = Cr$, $a_2'' = C$, and a_3'' found from (4.1.9), $a_3 = \frac{1}{2}(\rho_0^2 + 4R^2 r)$. This gives the familiar bound

$$\|U(t)\|_1^2 \leq \left(\frac{a_3''}{r} + a_2''\right)\exp(a_1'').$$

Case 3 Purely real diffusion, i.e. $\nu = 0$. Here we can find a point–wise bound which immediately yields an $L^{\infty}_{\Delta x}$ bound. Re–write (3.6.9) with $\nu = 0$ in component form:

$$\frac{dU_j}{dt} = RU_j + \delta^2 U_j - (1 + i\mu)|U_j|^2 U_j;$$

multiply by $\overline{U_j}$ and take the real part :

$$\frac{1}{2}\frac{d|U_j|^2}{dt} = R|U_j|^2 + \mathrm{Re}\left\{(\delta^2 U_j)\overline{U_j}\right\} - |U_j|^4.$$

Noting that

$$\begin{aligned} &2\mathrm{Re}\left\{(\delta^2 U_j)\overline{U_j}\right\} \\ &\quad = U_{j+1}\overline{U}_j + \overline{U}_{j+1}U_j + U_{j-1}\overline{U}_j + \overline{U}_{j-1}U_j - 4|U_j|^2, \\ &\quad = |U_{j+1}|^2 - 2|U_j|^2 + |U_{j-1}|^2 - |U_{j+1}U_j|^2 - |U_{j-1} - U_j|^2, \end{aligned}$$

we find that

$$\frac{1}{2}\frac{d|U_j|^2}{dt} \le R|U_j|^2 + \frac{1}{2}\delta^2|U_j|^2 - \frac{1}{2}|\delta_+ U_j|^2 - \frac{1}{2}|\delta_- U_j|^2 - |U_j|^4.$$

Hence,

$$\frac{d|U_j|^2}{dt} \le 2R|U_j|^2 + \delta^2|U_j|^2 - 2|U_j|^4.$$

Now if $|U_i|$ is a maximum the second term is negative, thus for maximum,

$$\frac{d|U_i|^2}{dt} \le 2R|U_i|^2 - 2|U_i|^4$$

and, provided

$$\max_{1 \le i \le J-1} |U_i|^2 < R$$

we have

$$\lim_{t \to \infty} |U|_{L^{\infty}_{\Delta x}} \le R.$$

Hence the theorem is proved.□.

We note that the bounds of Theorem 4.1.5 may be used in Theorem 4.1.4 and combined with the $L^2_{\Delta x}$ estimates give us $L^{\infty}_{\Delta x}$ bounds. This will not be pursued any further here.

4.2 Gevrey Class

In Section 3.4.2 the concept of Gevrey class and regularity was introduced and we stated that solutions to the complex Ginzburg–Landau equation were of a Gevrey class - see Theorem 3.4.6. We prove below that solutions to the semi–discrete system (3.6.9) are in a discrete class of regularity, which we defined in Section 3.7. In addition to being an interesting result, the extra regularity will be exploited to great advantage in Section (4.3).

Notation : To simplify forthcoming expressions we employ the following notation:

$$\sum_{k,\ell,m}{}' := \sum_{\substack{k-\ell+m=p+\eta J,\, \eta\in N \\ k,\ell,m\in[-\lceil J/2\rceil,\lfloor J/2\rfloor]}} \tag{4.2.1}$$

We shall now prove that solutions to the semi–discrete problem **SD**, (3.6.9) lie in a discrete Gevrey class.

Theorem 4.2.1 (Gevrey Regularity)

i) *Consider the semi-discrete complex Ginzburg–Landau equation* (3.6.9) *with initial condition* $U(0) = U^0 \in \mathcal{B}_1(\rho)$. *Then there exists* $T = T(\rho)$ *and* $\rho' = \sqrt{2(1+\rho^2)}$ *such that*

$$U(t) \in \mathcal{B}_t(\rho') \;\; \forall t \in (0,T].$$

ii) *Consider* (3.6.9) *with* $U^0 = U(0) \in \mathcal{B}_1(\rho_1)$. *Then there exists constant* $K = K(R,\nu,\mu) > 0$ *such that for*

$$\tau = \frac{3}{8K(1+\rho_1)}$$

we have

$$U(t) \in \mathcal{B}_\tau(\rho_2) \;\forall\; t \in (\tau,\infty)$$

where $\rho_2 = \sqrt{2(1+\rho_1)}$ *and* ρ_1 *is as in Theorem 4.1.2.*

Proof The method of proof follows that employed by Doelman and Titi [40] for (3.1.1) and Duan et al [42] for the generalized complex Ginzburg–Landau equation.

Let $U(t)$ be given by the Fourier series

$$U(t) = \sum_{k=-\lceil J/2\rceil}^{\lfloor J/2\rfloor} a_k(t)\psi_k$$

and define $V(t)$ by

$$V(t) \;:=\; e^{t\widetilde{A}^{\frac{1}{2}}}U(t) \;=\; \sum_{k=-J/2\rceil}^{J/2\rfloor} e^{t\widetilde{\lambda}_k^{\frac{1}{2}}}a_k(t)\psi_k \;, \qquad (4.2.2)$$

then we wish to find a bound for $|\widetilde{A}^{1/2}V|_{L^2_{\Delta x}}$. Now differentiate V with respect to t and substitute in the equation for U_t from (3.6.10)

$$\begin{aligned}
V_t &= \sum_{k=-J/2\rceil}^{J/2\rfloor} \widetilde{\lambda}_k^{\frac{1}{2}} e^{\widetilde{\lambda}_k^{\frac{1}{2}}t}a_k(t)\psi_k + \sum_{k=-J/2\rceil}^{J/2\rfloor} a_k'(t)e^{\widetilde{\lambda}_k^{\frac{1}{2}}t}\psi_k \\
&= \widetilde{A}^{\frac{1}{2}}V(t) + e^{t\widetilde{A}^{\frac{1}{2}}}U_t \\
&= \widetilde{A}^{\frac{1}{2}}V(t) + e^{t\widetilde{A}^{\frac{1}{2}}}\left\{\widetilde{R}U - (1+i\nu)\widetilde{A}U - (1+i\mu)G(|U|^2)U\right\} \\
&= \widetilde{A}^{\frac{1}{2}}V(t) + \widetilde{R}V - (1+i\nu)\widetilde{A}V - (1+i\mu)e^{t\widetilde{A}^{\frac{1}{2}}}G(|U|^2)U. \qquad (4.2.3)
\end{aligned}$$

Take the $L^2_{\Delta x}$ inner-product of (4.2.3) with $\widetilde{A}V$ and the real part :

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}|\widetilde{A}^{\frac{1}{2}}V|^2_{L^2_{\Delta x}} \;\le\;& \mathrm{Re}\left\{< \widetilde{A}^{1/2}V, \widetilde{A}V >\right\} + |\widetilde{R}||\widetilde{A}^{\frac{1}{2}}V|^2_{L^2_{\Delta x}} - |\widetilde{A}V|^2_{L^2_{\Delta x}} \\
& -\mathrm{Re}\left\{(1+i\mu)\left\langle e^{t\widetilde{A}^{\frac{1}{2}}}G(|U|^2)U, \widetilde{A}V\right\rangle\right\}.
\end{aligned}$$

Apply the generalised Cauchy–Schwarz inequality to get for all $\epsilon > 0$,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}|\widetilde{A}^{\frac{1}{2}}V|^2_{L^2_{\Delta x}} \;\le\;& (|\widetilde{R}| + \frac{1}{2\epsilon})|\widetilde{A}^{\frac{1}{2}}V|^2_{L^2_{\Delta x}} + \frac{\epsilon}{2}|\widetilde{A}V|^2_{L^2_{\Delta x}} - |\widetilde{A}V|^2_{L^2_{\Delta x}} \\
& -\mathrm{Re}\left\{(1+i\mu)\left\langle e^{t\widetilde{A}^{\frac{1}{2}}}G(|U|^2)U, \widetilde{A}V\right\rangle\right\}. \qquad (4.2.4)
\end{aligned}$$

Consider the non–linear term separately:

$$\begin{aligned}
e^{t\widetilde{A}^{1/2}}G(|U|^2)U &= e^{t\widetilde{A}^{1/2}}\left\{\sum_{k=-J/2\rceil}^{J/2\rfloor} a_k\psi_k\right\}\left\{\sum_{\ell=-J/2\rceil}^{J/2\rfloor} \overline{a_\ell}\overline{\psi_\ell}\right\}\left\{\sum_{m=-J/2\rceil}^{J/2\rfloor} a_m\psi_m\right\} \\
&= e^{t\widetilde{A}^{1/2}}\sum_{k,\ell,m}{}' a_k\overline{a_\ell}a_m\psi_p
\end{aligned}$$

Thus,

$$e^{t\widetilde{A}^{1/2}}G(|U|^2)U = \sum_{k,\ell,m}{}' a_k\overline{a_\ell}a_m e^{t\widetilde{\lambda}_p^{1/2}}\psi_p.$$

We let b_p denote the p^{th} Fourier coefficient for $e^{t\widetilde{A}^{1/2}}G(|U|^2)U$, so that

$$b_p = \left\langle e^{t\widetilde{A}^{1/2}}G(|U|^2)U, \psi_p\right\rangle.$$

Now by (3.7.4) the eigenvalue $\widetilde{\lambda}_p = 1 + \lambda_p$, where

$$\begin{aligned}\lambda_p^{\frac{1}{2}} &= \frac{2}{\Delta x}\sin(p\pi\Delta x)\\ &= \frac{2}{\Delta x}\sin((k-\ell+m-\eta J)\pi\Delta x),\end{aligned}$$

which by the trigonometric identity for $\sin(a+b)$ becomes

$$\begin{aligned}\lambda_p^{\frac{1}{2}} &\le \frac{2}{\Delta x}\sin(k\pi\Delta x) + \frac{2}{\Delta x}\sin(\ell\pi\Delta x) + \frac{2}{\Delta x}\sin(m\pi\Delta x)\\ &= \lambda_k^{\frac{1}{2}} + \lambda_\ell^{\frac{1}{2}} + \lambda_m^{\frac{1}{2}}\ .\end{aligned}$$

Thus,

$$\widetilde{\lambda}_p^{\frac{1}{2}} \le \widetilde{\lambda}_k^{\frac{1}{2}} + \widetilde{\lambda}_\ell^{\frac{1}{2}} + \widetilde{\lambda}_m^{\frac{1}{2}}$$

and

$$|b_p| \le \sum_{k,\ell,m}{}' e^{t\widetilde{\lambda}_k^{\frac{1}{2}}}|a_k|\, e^{t\widetilde{\lambda}_\ell^{\frac{1}{2}}}|a_\ell|\, e^{t\widetilde{\lambda}_m^{\frac{1}{2}}}|a_m|.$$

Now define

$$\widehat{V} = \sum_{k=-J/2\rceil}^{J/2\rfloor} e^{t\widetilde{\lambda}_k^{\frac{1}{2}}}|a_k|\psi_k \tag{4.2.5}$$

then

$$|\widehat{V}|_{L^2_{\Delta x}} = |V|_{L^2_{\Delta x}}\ ; |\widetilde{A}^{\frac{1}{2}}\widehat{V}|_{L^2_{\Delta x}} = |\widetilde{A}^{\frac{1}{2}}V|_{L^2_{\Delta x}}\ ;\ |\widetilde{A}\widehat{V}|_{L^2_{\Delta x}} = |\widetilde{A}V|_{L^2_{\Delta x}}.$$

Furthermore, the Fourier coefficient of $G(|\widehat{V}|^2)\widehat{V}$, c_p, $(p = -J/2\rceil, \cdots J/2\rfloor)$, is given by

$$c_p = \sum_{k,\ell,m}{}' e^{t\widetilde{\lambda}_k^{\frac{1}{2}}}|a_k|\, e^{t\widetilde{\lambda}_\ell^{\frac{1}{2}}}|a_\ell|\, e^{t\widetilde{\lambda}_m^{\frac{1}{2}}}|a_m|.$$

Therefore

$$\begin{aligned}&\mathrm{Re}\left\{(1+i\mu)\left\langle e^{t\widetilde{A}^{\frac{1}{2}}}G(|U|^2)U, \widetilde{A}V\right\rangle\right\}\\ &= \mathrm{Re}\left\{(1+i\mu)\sum_{j=0}^{J-1}\Delta x\left(\sum_{p=-J/2\rceil}^{J/2\rfloor} b_p e^{2\pi i p j\Delta x}\right)\left(\sum_{q=-J/2\rceil}^{J/2\rfloor}\widetilde{\lambda}_q e^{t\widetilde{\lambda}_q^{\frac{1}{2}}}\overline{a_q}e^{-2\pi i q j\Delta x}\right)\right\}\\ &= \mathrm{Re}\left\{(1+i\mu)\sum_{j=0}^{J-1}\Delta x\sum_{k,\ell,m}{}' b_p\overline{a_p}\widetilde{\lambda}_p e^{t\widetilde{\lambda}_q^{1/2}}\right\}\\ &\le (1+\mu^2)^{\frac{1}{2}}\left\{\sum_{j=0}^{J-1}\Delta x\sum_{k,\ell,m}{}' |b_p||a_p|\widetilde{\lambda}_p e^{t\widetilde{\lambda}_q^{1/2}}\right\}\\ &\le (1+\mu^2)^{\frac{1}{2}}\left\{\sum_{j=0}^{J-1}\Delta x\sum_{k,\ell,m}{}' c_p|a_p|\widetilde{\lambda}_p e^{t\widetilde{\lambda}_q^{1/2}}\right\}\end{aligned}$$

$$
\begin{aligned}
&\le (1+\mu^2)^{\frac{1}{2}} \left\{ \sum_{j=0}^{J-1} \Delta x \left(\sum_{p=-J/2\rceil}^{J/2\rfloor} c_p e^{2\pi i p j \Delta x} \right) \left(\sum_{q=-J/2\rceil}^{J/2\rfloor} \widetilde{\lambda}_q e^{t\widetilde{\lambda}_q^{\frac{1}{2}}} |a_q| e^{-2\pi i q j \Delta x} \right) \right\} \\
&= (1+\mu^2)^{1/2} \left\langle G(|\widehat{V}|)\widehat{V}, \widetilde{A}^{1/2}\widehat{V} \right\rangle \qquad (4.2.6)
\end{aligned}
$$

If we expand the inner–product in (4.2.6) and then apply Cauchy–Schwarz and Lemma 3.7.4 we find

$$
\begin{aligned}
&\mathrm{Re}\left\{ (1+i\mu) < e^{t\widetilde{A}^{\frac{1}{2}}} G(|U|^2)U, \widetilde{A}V > \right\} \\
&\le (1+\mu^2)^{\frac{1}{2}} \sum_{j=0}^{J-1} \Delta x |\widehat{V}_j|^3 \left| \overline{\widehat{V}_j} + \frac{\delta^2 \overline{\widehat{V}_j}}{\Delta x^2} \right| \\
&\le (1+\mu^2)^{\frac{1}{2}} \left\{ \sum_{j=0}^{J-1} \Delta x |\widehat{V}_j|^6 \right\}^{1/2} \left\{ \sum_{j=0}^{J-1} \Delta x |\overline{\widehat{V}_j} + \frac{\delta^2 \overline{\widehat{V}_j}}{\Delta x^2}|^2 \right\}^{1/2} \\
&= (1+\mu^2)^{\frac{1}{2}} |\widehat{V}|^3_{L^6_{\Delta x}} |\widetilde{A}V|_{L^2_{\Delta x}}. \qquad (4.2.7)
\end{aligned}
$$

By the discrete Gagliardo–Nirenberg inequality (Lemma 3.7.9 with $p=6$, $q=2$),

$$
\begin{aligned}
|\widehat{V}|_{L^6_{\Delta x}} &\le 3^{1/6} |\widehat{V}|^{1/3}_{L^2_{\Delta x}} \|\widehat{V}\|^{2/3}_{H^1_{\Delta x}} \\
&= 3^{1/6} |V|^{1/3}_{L^2_{\Delta x}} |\widetilde{A}^{1/2} V|^{2/3}_{L^2_{\Delta x}}. \qquad (4.2.8)
\end{aligned}
$$

So after applying the generalized Cauchy–Schwarz inequality we may re–write (4.2.7) as

$$
\begin{aligned}
&\mathrm{Re}\left\{ (1+i\mu) < e^{t\widetilde{A}^{\frac{1}{2}}} G(|U|^2)U, \widetilde{A}V > \right\} \\
&\le (1+\mu^2)^{1/2} \left\{ \frac{1}{2\epsilon} |V|^2_{L^2_{\Delta x}} |\widetilde{A}^{1/2} V|^4_{L^2_{\Delta x}} + 3^{1/3}\frac{\epsilon}{2} |\widetilde{A}V|^2_{L^2_{\Delta x}} \right\}. \qquad (4.2.9)
\end{aligned}
$$

Finally we return to the full equation (4.2.4), and substitute in (4.2.9) to get :

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt} |\widetilde{A}^{\frac{1}{2}} V|^2_{L^2_{\Delta x}} &\le (|\widetilde{R}| + \frac{1}{2\epsilon}) |\widetilde{A}^{\frac{1}{2}} V|^2_{L^2_{\Delta x}} + (\frac{\epsilon}{2} - 1) |\widetilde{A}V|^2_{L^2_{\Delta x}} \\
&\quad + (1+\mu^2)^{1/2} \left\{ \frac{1}{2\epsilon} |V|^2_{L^2_{\Delta x}} |\widetilde{A}^{1/2} V|^4_{L^2_{\Delta x}} + 3^{1/3}\frac{\epsilon}{2} |\widetilde{A}V|^2_{L^2_{\Delta x}} \right\} \\
&= (|\widetilde{R}| + \frac{1}{2\epsilon}) |\widetilde{A}^{\frac{1}{2}} V|^2_{L^2_{\Delta x}} + (1+\mu^2)^{1/2} \frac{1}{2\epsilon} |\widetilde{A}^{1/2} V|^6_{L^2_{\Delta x}} \\
&\quad + \left(\frac{\epsilon}{2}(1 + 3^{1/3}(1+\mu^2)^{1/2}) - 1 \right) |\widetilde{A}V|^2_{L^2_{\Delta x}} \qquad (4.2.10)
\end{aligned}
$$

where (4.2.10) is found by noting that $|V|_{L^2_{\Delta x}} \le |\widetilde{A}^{1/2} V|_{L^2_{\Delta x}}$.

We now make our choice of ϵ,

$$\epsilon < \frac{2}{1+3^{1/3}(1+\mu^2)^{\frac{1}{2}}},$$

to dispose of the $|\widetilde{A}V|^2_{L^2_{\Delta x}}$ term. Thus,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\widetilde{A}^{\frac{1}{2}}V|^2_{L^2_{\Delta x}} &\leq (|\widetilde{R}|+\frac{1}{2\epsilon})|\widetilde{A}^{\frac{1}{2}}V|^2_{L^2_{\Delta x}} + (1+\mu^2)^{1/2}\frac{1}{2\epsilon}|\widetilde{A}^{1/2}V|^6_{L^2_{\Delta x}} \\ &= K_1(R,\nu,\mu)|\widetilde{A}^{\frac{1}{2}}V|^2_{L^2_{\Delta x}} + K_3(R,\nu,\mu)|\widetilde{A}^{\frac{1}{2}}V|^6_{L^2_{\Delta x}}. \qquad (4.2.11)\end{aligned}$$

Let $Y(t) = 1 + |\widetilde{A}^{\frac{1}{2}}V(t)|^2_{L^2_{\Delta x}}$ then we can re-write (4.2.11) as

$$\frac{d}{dt}Y \leq KY^3 \qquad (4.2.12)$$

with, for example,

$$K = K(R,\nu,\mu) = \frac{1}{3}K_1 + K_3.$$

Thus we find that for $t < \frac{1}{2K(Y^0)^2}$

$$Y(t) \leq \frac{Y^0}{\sqrt{1-2Kt(Y^0)^2}} \qquad (4.2.13)$$

and so for

$$0 \leq t \leq T := \frac{3}{8K(1+\|U^0\|^2_{H^1_{\Delta x}})^2}, \qquad (4.2.14)$$

we find from (4.2.13) that

$$\begin{aligned}|\widetilde{A}^{1/2}e^{t\widetilde{A}^{1/2}}U(t)|^2_{L^2_{\Delta x}} &\leq Y(T) & (4.2.15)\\ &= \frac{Y^0}{\sqrt{1-3/4}} & (4.2.16)\\ &= 2(1+\|U^0\|^2_{H^1_{\Delta x}}). & (4.2.17)\end{aligned}$$

Hence part *i*) is proved with $\rho' := \sqrt{2(1+\|U^0\|^2_{H^1_{\Delta x}})}$.

ii) Let t_1 and ρ_1 be as in Theorem 4.1.3 – so that

$$\|U(t)\|_{H^1_{\Delta x}} \leq \rho_1 \; \forall t > t_1$$

and ρ_1 is independent of Δx. Then the analysis of *i*) can be repeated for all $t > t_1$ and so for

$$\tau = \frac{3}{8K(1+\rho_1)}$$

we find

$$|\widetilde{A}^{1/2} e^{\tau \widetilde{A}^{1/2}} U|^2_{L^2_{\Delta x}} \leq 2(1+\rho_1^2).$$

Thus *ii*) is proved with $\rho_2 = \sqrt{2(1+\rho_1^2)}$.□

Note

• Suppose we were given $U^0 = U(0) \in \mathcal{B}_1(\rho)$ where $\rho > \rho_1$, and so our initial data lay outside the $H^1_{\Delta x}$ absorbing set. Then we may prove that $\exists\ \sigma, \rho_3$ such that for all $t > 0$

$$U(t) \in \mathcal{B}_\sigma(\rho_3).$$

This is achieved by proving from (4.1.13) and (4.1.7) the existence of a constant M such that

$$|\widetilde{A}^{\frac{1}{2}} U|^2_{L^2_{\Delta x}} \leq C(t_1), \quad t \in [0, t_1]$$

and then repeating the proof of Theorem 4.2.1 *i*) until $t > t_1$. For $t > t_1$ we have exactly Theorem 4.2.1 part *ii*).

The previous theorem states that given any initial data $U^0 \in \mathcal{B}_1(\rho)$ we have a Gevrey ball $\mathcal{B}_{G_{\tau,\Delta x}}(\rho_2)$ which is absorbing. This leads us to the following corollary.

Corollary 4.2.1 *Let τ and ρ_2 be as in Theorem 4.2.1 part ii). Then the global attractor $\mathcal{A}_{\Delta x}$ of Theorem 4.1.4 is given by*

$$\mathcal{A}_{\Delta x} = \omega(\mathcal{B}_\tau(\rho_2)) \tag{4.2.18}$$

where ρ_2 is independent of Δx.

Proof From Theorem 4.1.4 we have that

$$\mathcal{A} = \omega(\mathcal{B}_1(\rho_1)),$$

and Theorem 4.2.1 *ii*) yields the existence of $T > 0$ such that

$$S(t)\mathcal{B}_1(\rho_1) \subset \mathcal{B}_\tau(\rho_2);$$

and the result is immediate.□

4.3 Convergence of Attractors

In this section we prove upper–semicontinuity of the approximate global attractor $\mathcal{A}_{\Delta x}$ (established in Theorem 4.1.5) to the continuous global attractor $\mathcal{A}$. We shall examine upper–semicontinuity using the notion of semi–distance defined in Definition 1.2.1.

In the proof there will be two different limiting processes to consider: the limit as $\Delta x \rightarrow 0$ and the limit as $t \rightarrow \infty$. Standard error estimates alone are not enough to prove a convergence result as they are of the form

$$||U_{\text{true}} - U_{\text{numerical}}|| \leq C_1 \Delta x^p e^{C_2 T} \quad \forall\, t \in (0, T], \tag{4.3.1}$$

and give no time asymptotic information. In essence we prove the result over any *finite* time interval and use *induction* to extend to the infinite time interval. Over the finite interval we apply the standard type of estimate. It is the attracting property which allows us to perform the induction step. This basic method of proof was introduced by Hale, Lin and Raugel [64].

▷ To proceed we require an error estimate of the form (4.3.1) and the method for finding such an estimate depends on the regularity of the solutions. However, a priori it is not necessarily known if the solution is sufficiently regular (to apply Taylor's theorem), in which case non–smooth error estimates are required. These types of estimates may be found in a variety of ways, and one possible approach is to extend the absorbing set analysis from which we obtained H^1 regularity of solutions. However extending that kind of analysis presents many technical difficulties see for example [102, 8, 9]. For finite element methods it is common to use the smoothing property of the linear semigroup generated by $\widetilde{A}$, to find non–smooth error estimates (see for example Larsson [89] Wahlbin, Bramble et al [130, 17] or [52]). This leads to an error estimate with a singularity in t of the form

$$||U_{\text{true}} - U_{\text{numerical}}||_{H^1} \leq C_1 \Delta x^p t^{-\frac{1}{2}} \quad \forall t \in (0, T]$$

for an initial condition $U^0 \in L^2$. This approach also applies to the finite difference setting; however the singularity in t from the smoothing of the linear semi–group causes certain technical details when trying to apply Gronwall's lemma.

For the *linear* problem Strikwerda [124, Secs 10.3-10.4] proves error estimates for finite differences of the form

$$||U_{\text{true}} - U_{\text{numerical}}||_{L^2} \leq C_1 \Delta x^{\sigma/2} ||U^0||_{H^\sigma} \quad \forall t \in (0,T], \sigma < 4$$

which could possibly be adapted to the non–linear case. For the linear parabolic case he obtains bounds of the form

$$||U_{\text{true}} - U_{\text{numerical}}||_{L^2} \leq C_1(1 + t^{-3/2})\Delta x^2 ||U^0||_{L^2} \quad \forall t \in (0,T]$$

The results of Strikwerda are very similar to those in the lecture notes of Brenner et al [19]. They present non–smooth error estimates for finite difference approximations to linear equations of the types above but set in Besov spaces.

$\triangleright$ When the solutions are smooth, a smooth error analysis may be performed (e.g. as in [108] or [124, Sec. 10.1]), and the error is essentially estimated from Taylor's theorem.

Yin Yan [137] proves upper–semicontinuity of the global attractors for finite difference approximations to the Navier–Stokes equations. As far as we are aware this is the only other upper–semicontinuity result for finite difference approximations to partial differential equations. Yan proves the result by a piecewise linear interpolation at the grid points to set the analysis in the continuous space L^2 and derives non–smooth data error estimates.

In contrast to these results we use the Gevrey regularity of the solution and semi–discrete solution to prove upper-semicontinuity in $H^1_{\Delta x}$ with a bound independent of Δx.

The following theorem bounds the error in the $H^1_{\Delta x}$ norm between continuous and discrete trajectories inside a Gevrey ball.

Theorem 4.3.1 *Given $\tau, \rho > 0$ and an initial condition for (3.6.9) $U^0 \in B_\tau(\rho)$ suppose we have found V^0 from Lemma 3.7.16 so that $V^0 \in G_\sigma$ and $P_{\Delta x} V^0 = U^0$ where $\sigma \in (0, \frac{2}{\pi}\tau)$.*

Then, for any $T > 0$, $\exists C = C(T, \rho) > 0$:

$$||S_{\Delta x}(t)U^0 - P_{\Delta x} S(t) V^0||_{H^1_{\Delta x}} < \bar{C}\Delta x^2 \quad \forall\, t \in (0,T] \tag{4.3.2}$$

where $S_{\Delta x}$ is the semi-group of Theorem 4.1.1 for the semi-discrete problem (3.6.9) *and $S(t)$ is the semi-group of Theorem 3.4.2 for the continuous problem* (3.1.1).

Proof Using the Gevrey class of our initial data, we follow a standard smooth error analysis proof.

The evolution $S_{\Delta x}(t)U^0 = U(t)$ satisfies the semi-discrete equation (3.6.10)

$$\frac{d}{dt}U = -(1+i\nu)\tilde{A}U(t) + F(U(t))$$

and $P_{\Delta x}S(t)V_0 = P_{\Delta x}V(t) = V(t)$ satisfies :

$$\begin{aligned} \frac{d}{dt}V &= \tilde{R}V(t) - (1+i\nu)\tilde{A}V(t) - (1+i\mu)G(|V(t)|^2)V(t) + \eta(t) \\ &= -(1+i\nu)\tilde{A}V(t) + F(V(t)) + \eta(t) \end{aligned} \tag{4.3.3}$$

where $\eta(t)$ is the truncation error.

Since $V(0)$ is in Gevrey class G_σ by Duan et al [42, Theorem 2] we have that $V(t) \in H^\alpha \ \forall\ t > 0$ and $\forall\ \alpha > 0$. Now in one space dimension ($p = 1$) we have that $C^4 \supset H^5$ (see for example [81] or [104]) and hence we have the required regularity to apply Taylor's theorem and the mean value theorem to estimate the truncation error $\eta(t)$

$$\eta(t) = \frac{\Delta x^2}{12}\left(\frac{\partial^4 V}{\partial x^4}|_{\theta_0}, \frac{\partial^4 V}{\partial x^4}|_{\theta_1+\Delta x}, \cdots, \frac{\partial^4 V}{\partial x^4}|_{\theta_{J-1}+(J-1)\Delta x}\right)$$

where $\theta_i \in (0, \Delta x)$ and $\Delta x < 1$. Let $e(t) = V(t) - U(t)$. Then the error $e(t)$ satisfies :-

$$\frac{d}{dt}e = -(1+i\nu)\tilde{A}e + F(V) - F(U) + \eta(t). \tag{4.3.4}$$

Let $E_{\Delta x}(t)$ denote the linear semi-group generated by $(1+i\nu)\tilde{A}$. Then by applying Duhamel's principle equation (4.3.4) becomes

$$e(t) = E_{\Delta x}(t)e(0) + \int_0^t E_{\Delta x}(t-s)\left(F(V(s)) - F(U(s))\right)ds + \frac{\Delta x^2}{12}\int_0^t E_{\Delta x}(t-s)\eta(t)\,ds. \tag{4.3.5}$$

Now taking the $H^1_{\Delta x}$ norm of (4.3.5), using the smoothing property of the linear semi-group $E(t)$ given in Theorem 3.2.9 and that $e(0) = 0$ we get

$$e(t) \le \int_0^t (t-s)^{-\frac{1}{2}}|F(V(s)) - F(U(s))|_{L^2_{\Delta x}}ds + \frac{\Delta x^2}{12}\int_0^t (t-s)^{-\frac{1}{2}}\,|\eta(t)|_{L^2_{\Delta x}}\,ds. \tag{4.3.6}$$

The final term in (4.3.6) is dealt with by noting that since $V \in H^5$ we can bound $|\eta(t)|_{L^2_{\Delta x}}$ uniformly over any finite time interval $[0,T]$. For the second term in (4.3.6), recall Corollary 3.7.1 and note that since $U^0, V^0 \in \mathcal{B}_{G_{r,\Delta x}}(\rho)$, $U(t), V(t) \in \mathcal{B}_1(\rho)$, $\rho > 0$, independent of Δx. So by Lemma 3.7.8 we have uniform bounds on the $L^\infty_{\Delta x}$ norm.

$$
\begin{aligned}
|F(U(t)) - F(V(t))|_{L^2_{\Delta x}} &\leq \left(|\tilde{R}| + (1+\mu^2)^{\frac{1}{2}}(|U(t)|^2_{L^\infty_{\Delta x}} + |V(t)|^2_{L^\infty_{\Delta x}})\right) |U(t) - V(t)|^2_{L^2_{\Delta x}} \\
&\leq C_1(T)\|U - V\|^2_{H^1_{\Delta x}}. \qquad (4.3.7)
\end{aligned}
$$

Therefore (4.3.6) becomes

$$
\begin{aligned}
\|e(t)\|_{H^1_{\Delta x}} &\leq C_1 \int_0^t (t-s)^{-\frac{1}{2}} \|e(s)\|_{H^1_{\Delta x}} ds + C_0 \Delta x^2 \int_0^t (t-s)^{-\frac{1}{2}} ds \\
&\leq C_1 \int_0^t (t-s)^{-\frac{1}{2}} \|e(s)\|_{H^1_{\Delta x}} ds + C_0 \Delta x^2 t^{1/2}. \qquad (4.3.8)
\end{aligned}
$$

By an application of Gronwall's lemma 3.4.4 we obtain

$$
\|e(t)\|_{H^1_{\Delta x}} \leq C(T)\Delta x^2 \qquad (4.3.9)
$$

for $t \in (0,T]$.□

We are now in a position to prove our main theorem of this section on the upper–semicontinuity of the attractors.

Theorem 4.3.2 (Upper–Semicontinuity) *Let $\mathcal{A}$ denote the global attractor for the continuous system as in section 3.4.1. Let $\mathcal{A}_{\Delta x}$ denote the global attractor of Theorem 4.1.4 for the discrete system with $0 < \Delta x < 1$. Then*

$$
dist_{H^1_{\Delta x}}(\mathcal{A}_{\Delta x}, P_{\Delta x}\mathcal{A}) \rightarrow 0 \text{ as } \quad \Delta x \rightarrow 0. \qquad (4.3.10)
$$

Proof

Recall the result of Corollary 4.2.1, namely there exists ρ_2 independent of Δx and $\tau > 0$ such that

$$
\mathcal{A}_{\Delta x} = \omega\left(\mathcal{B}_{G_{r,\Delta x}}(\rho_2)\right) = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} S_{\Delta x}(t)\mathcal{B}_{G_{r,\Delta x}}(\rho_2)}. \qquad (4.3.11)
$$

Thus it is sufficient to prove that $\forall\, \epsilon > 0$, $\exists\, \Delta x_0$, T^* *s.t.*

$$
S_{\Delta x}(t)\mathcal{B}_{G_{r,\Delta x}}(\rho_2) \in N(P_{\Delta x}\mathcal{A}, \epsilon) \;\; \forall\, t > T^*,\; \forall\, \Delta x < \Delta x_0. \qquad (4.3.12)
$$

If we can prove that (4.3.12) holds, then by (4.3.11) we have that

$$\mathcal{A}_{\Delta x} \in N(P_{\Delta x}\mathcal{A}, \epsilon) \tag{4.3.13}$$

and the theorem is proved.

We now proceed to prove (4.3.12) by induction. Let $\epsilon > 0$ be given.
First note that by Lemma 3.7.16 for every $U^0 \in \mathcal{B}_\tau(\rho_2)$

$$\exists\ \sigma > 0 \text{ and } V^0 \in G_\sigma \ s.t.\ P_{\Delta x}V^0 = U^0.$$

By the attracting property of $\mathcal{A}$, Theorem (4.1.4), $\exists\ T = T(\epsilon, \rho_2) > 0$ $s.t.$

$$\text{dist}_{H^1}(S(t)V(0), \mathcal{A}) \leq \frac{\epsilon}{2} \quad \forall\, t \geq T$$

and so by Lemma 3.7.3

$$\text{dist}_{H^1_{\Delta x}}(P_{\Delta x}S(t)V(0), P_{\Delta x}\mathcal{A}) \leq \frac{\epsilon}{2} \quad \forall\, t \geq T. \tag{4.3.14}$$

Furthermore by the standard error estimate of Theorem (4.3.1) for all $U^0 \in \mathcal{B}_\tau(\rho_2)$ and $V^0 \in G_\sigma$ such that

$$\|S_{\Delta x}(t)U(0) - P_{\Delta x}S(t)V(0)\|_{H^1_{\Delta x}} \leq \frac{\epsilon}{2} \quad \forall\, t \in (0, 2T] \tag{4.3.15}$$

provided $\Delta x^2 \leq \Delta x_0^2 := \frac{\epsilon}{2C(2T)}$.

Combining the two properties (4.3.14) and (4.3.15) we get that $\forall U^0 \in \mathcal{B}_\tau(\rho_2)$,

$$\begin{aligned}
&\text{dist}_{H^1_{\Delta x}}(S_{\Delta x}(t)U^0, P_{\Delta x}\mathcal{A}) \\
&= \inf_{U\in\mathcal{A}} \|S_{\Delta x}(t)U^0 - P_{\Delta x}U\|_{H^1_{\Delta x}} \\
&\leq \inf_{U\in\mathcal{A}} \|P_{\Delta x}S(t)V^0 - P_{\Delta x}U\|_{H^1_{\Delta x}} + \|S_{\Delta x}(t)U^0 - P_{\Delta x}S(t)V^0\|_{H^1_{\Delta x}} \\
&= \text{dist}_{H^1_{\Delta x}}(P_{\Delta x}S(t)V^0, P_{\Delta x}\mathcal{A}) + \|S_{\Delta x}(t)U^0 - P_{\Delta x}S(t)V^0\|_{H^1_{\Delta x}} \\
&\leq \epsilon/2 + \epsilon/2 \qquad \forall t \in [T, 2T].
\end{aligned}$$

and so

$$U(t) \in N(\mathcal{A}, \epsilon) \quad \forall\, t \in [T, 2T].$$

To complete the inductive step note that

$$S_{\Delta x}(t)\mathcal{B}_\tau(\rho_2) \subset \mathcal{B}_\tau(\rho_2), \ \forall t > 0 \tag{4.3.16}$$

and so in particular $S_{\Delta x}(T)\mathcal{B}_\tau(\rho_2) \subset \mathcal{B}_\tau(\rho_2)$.

Thus the above argument may be repeated for $t \in [2T, 3T]$, yielding $\forall U^0 \in \mathcal{B}_\tau(\rho_2)$

$$U(t) \in N(\mathcal{A}, \epsilon) \ \forall t \in [T, 3T].$$

By property (4.3.16) we may repeat the argument again for the intervals $[3T, 4T], [4T, 5T] \cdots$

Hence (4.3.12) hold by induction and the theorem is proved.□

4.4 The Fully Discrete Finite Difference Schemes

We now consider fully discrete approximations to the continuous complex Ginzburg–Landau equation **CGL**, in particular the fully implicit scheme **DI** and the mixed scheme **DEI**. For each scheme we aim to prove discrete analogues of **C1-C4**, **DI1-DI4** and **DEI1-DEI4**.

To ensure that the fully discrete dynamical systems have the desired dynamical properties it is necessary to enforce restrictions on the time step Δt. Ideally we seek restrictions on Δt independent of the spatial step size Δx and the initial condition U^0. For the scheme **DI** this was achieved, however for the mixed scheme **DEI** we present results with Δt dependent on Δx.

We also considered the Θ type method given by

$$\begin{aligned}\frac{U^{n+1} - U^n}{\Delta t} &= R\left(\Theta U^{n+1} + (1-\Theta)U^n\right) - (1+i\nu)M^{-1}A\left(\Theta U^{n+1} + (1-\Theta)U^n\right) \\ &\quad - (1+i\mu)G\left(\left|\left(\Theta U^{n+1} + (1-\Theta)U^n\right)\right|^2\right)\left(\Theta U^{n+1} + (1-\Theta)U^n\right),\end{aligned} \tag{4.4.1}$$

where $\Theta \in [0, 1]$ and we are given some $U^0 \in L^2_{L^2_{\Delta x}}$.

For $\Theta > 1/2$ we are able to prove the existence of $L^2_{\Delta x}$ and $H^1_{\Delta x}$ absorbing balls with radius ρ_0^Θ and ρ_1^Θ respectively; with both ρ_0^Θ and ρ_1^Θ independent of Δx, Δt and the initial condition U^0. For the absorbing ball $\mathcal{B}_0(\rho_0^\Theta)$ we require that the time step Δt be restricted so that

$$\Delta t < \frac{2\Theta - 1}{6R\Theta(1-\Theta)},$$

and for the absorbing ball $\mathcal{B}_1(\rho_0^\Theta)$ we require that

$$\Delta t < \Delta t_1(R, \mu, \Theta, \rho_0^\Theta).$$

We shall not present the analysis here since it is analogous to the analysis for fully implicit scheme (3.6.12) which we present below.

4.4.1 Fully Implicit Scheme DI

The first scheme we consider is the fully implicit scheme **DI** given by equation (3.6.12) with U^0 bounded in $L^2_{\Delta x}$ independently of Δx. The aim is to prove for the scheme the discrete analogues **DI1-DI4** of **C1-C4**.

DI1 Existence and uniqueness of a solution U^n given U^0,

DI2 Existence of an absorbing ball, $\mathcal{B}_0(\rho_0)$, with ρ_0 independent of Δx,

DI3 Existence of an absorbing ball, $\mathcal{B}_1(\rho_1)$, with ρ_1 independent of Δx,

DI4 Existence of a global attractor.

For convenience we shall assume that we are given a semi–group $S^n : L^2_{\Delta x} \to L^2_{\Delta x}$ for (3.6.12) and will first establish **DI2** and **DI3** - and then return to **DI1** and **DI4**.

Theorem 4.4.1 (DI2) *Given $U^0 \in L^2_{\Delta x}$ suppose that $\forall n > 0$ there exists a solution $U^n \in L^2_{\Delta x}$ of (3.6.12). Then there exists $\rho_0 = \rho_0(R) > 0$, independent of Δx, such that $\mathcal{B}_0(\rho_0)$ is absorbing and positively invariant.*

That is for $B \subset \mathcal{B}_0(\rho)$ $\exists$ $n_0 = n_0(\rho, \rho_0)$ such that

$$S^n_{\Delta x} B \subset \mathcal{B}_0(\rho_0) \; \forall n > n_0.$$

Proof Take the $L^2_{\Delta x}$ inner–product of (3.6.12) with U^{n+1},

$$\begin{aligned} |U^{n+1}|^2_{L^2_{\Delta x}} - < U^n, U^{n+1} > &= R\Delta t |U^{n+1}|^2_{L^2_{\Delta x}} - \Delta t(1+i\nu) < M^{-1}AU^{n+1}, U^{n+1} > \\ &\quad -\Delta t(1+i\mu) < G(|U^{n+1}|^2)U^{n+1}, U^{n+1} > . \end{aligned}$$

Then take the real part, complete the square and rearrange to get

$$|U^{n+1}|^2_{L^2_{\Delta x}} \le |U^n|^2_{L^2_{\Delta x}} + 2R\Delta t |U^{n+1}|^2_{L^2_{\Delta x}} - 2\Delta t \|U^{n+1}\|^2_1 - 2\Delta t |U^{n+1}|^4_{L^4_{\Delta x}}. \qquad (4.4.2)$$

Let us consider the three possible cases for R.

1. If $R < 0$ then equation (4.4.2) becomes

$$|U^{n+1}|^2_{L^2_{\Delta x}} \leq |U^n|^2_{L^2_{\Delta x}} + 2R\Delta t|U^{n+1}|^2_{L^2_{\Delta x}};$$

thus

$$|U^{n+1}|^2_{L^2_{\Delta x}} \leq \frac{1}{1-2R\Delta t}|U^n|^2_{L^2_{\Delta x}}$$

and since $(1 - 2R\Delta t) > 1$, we see that $|U^n|^2_{L^2_{\Delta x}} \to 0$ as $n \to \infty$. Therefore, for any $\epsilon > 0$, we may take $\rho_0 = \epsilon$.

2. If $R = 0$ then using Lemma 3.7.7 with $p = 2$ on the non–linear term in (4.4.2)

$$|U^{n+1}|^2_{L^2_{\Delta x}} \leq |U^n|^2_{L^2_{\Delta x}} - 2\Delta t|U^{n+1}|^4_2.$$

The sequence $\left(|U^n|^2_{L^2_{\Delta x}}\right)$ is strictly monotonically decreasing unless $\exists\, n^*$ such that $|U^{n^*}|_{L^2_{\Delta x}} = 0$. In which case

$$\forall n > n^*, \quad |U^n|^2_{L^2_{\Delta x}} = 0.$$

Therefore 0 is a lower bound and the only possible limit

$$\Longrightarrow |U^{n+1}|_{L^2_{\Delta x}} \to 0 \text{ as } n \to \infty.$$

Hence we may take $\rho_0 = \epsilon$ for any $\epsilon > 0$.

3. If $R > 0$ then using the inequality (4.1.5) to bound half the $L^4_{\Delta x}$ term in equation (4.4.2) we get

$$\begin{aligned}
&|U^{n+1}|^2_{L^2_{\Delta x}}(1 - 2R\Delta t) \\
&\leq \ |U^n|^2_{L^2_{\Delta x}} - 2\Delta t\|U^{n+1}\|^2_1 - \Delta t|U^{n+1}|^4_{L^4_{\Delta x}} - 4R\Delta t|U^{n+1}|^2_{L^2_{\Delta x}} + 4R^2\Delta t \\
&\leq \ |U^n|^2_{L^2_{\Delta x}} - 2\Delta t\|U^{n+1}\|^2_1 - \Delta t|U^{n+1}|^4_{L^4_{\Delta x}} + 4R^2\Delta t. \qquad (4.4.3)
\end{aligned}$$

Thus,

$$|U^{n+1}|^2_{L^2_{\Delta x}}(1 + 2R\Delta t) \leq |U^n|^2_{L^2_{\Delta x}} + 4R^2\Delta t$$

to which we can apply the standard discrete Gronwall argument,

$$\begin{aligned}
|U^{n+1}|^2_{L^2_{\Delta x}} &\leq \left(\frac{1}{1+2R\Delta t}\right)^n |U^0|^2_{L^2_{\Delta x}} + 4R^2\Delta t\sum_{k=1}^{n}\frac{1}{(1+2R\Delta t)^k} \\
&= \frac{1}{(1+2R\Delta t)^n}|U^0|^2_{L^2_{\Delta x}} + 4R^2\Delta t\left\{\frac{1-(1+2R^2\Delta t)^{-n}}{1+2R\Delta t-1}\right\} \\
&= \frac{1}{(1+2R\Delta t)^n}|U^0|^2_{L^2_{\Delta x}} + 2R + \frac{2R}{(1+2R\Delta t)^n}. \qquad (4.4.4)
\end{aligned}$$

Since $(1+2R\Delta t) > 1$ we find

$$\lim_{N\to\infty} |U^{N+1}|^2_{L^2_{\Delta x}} = 2R;$$

thus if we choose

$$\rho_0^2 > 2R, \tag{4.4.5}$$

there exists $n_0(|U^0|_{L^2_{\Delta x}})$ such that the ball $\mathcal{B}_0(\rho_0)$ is absorbing and ρ_0 is independent of Δx.□

Notes

• We can sum (4.4.3) over n from k_0 to $N+k_0$ for $k_0 \geq n_0$ to obtain a discrete analogue of the integral bound (4.1.9):

$$\sum_{n=k_0}^{N+k_0} (|U^{n+1}|^2_{L^2_{\Delta x}} - |U^n|^2_{L^2_{\Delta x}} + 2R\Delta t|U^{n+1}|^2_{L^2_{\Delta x}} + 2\Delta t\|U^{n+1}\|_1^2 + \Delta t|U^{n+1}|^4_{L^4_{\Delta x}}) \leq 4\Delta t R^2 N.$$

Thus,

$$\sum_{n=k_0}^{N+k_0} (2R\Delta t|U^{n+1}|^2_{L^2_{\Delta x}} + 2\Delta t\|U^{n+1}\|_1^2 + \Delta t|U^{n+1}|^4_{L^4_{\Delta x}}) \leq 4\Delta t R^2 N - |U^{N+k_0+1}|^2_{L^2_{\Delta x}} + |U^{k_0}|^2_{L^2_{\Delta x}}. \tag{4.4.6}$$

By Theorem (4.4.1) for $k_0 > n_0$ we can bound the right–hand side of (4.4.6)

$$\sum_{n=k_0}^{N+k_0} (2R\Delta t|U^{n+1}|^2_{L^2_{\Delta x}} + 2\Delta t\|U^{n+1}\|_1^2 + \Delta t|U^{n+1}|^4_{L^4_{\Delta x}}) \leq 4rR^2 + \rho_0^2 \tag{4.4.7}$$

where $r = N\Delta t$.

• The condition that $\rho_0 > 2R$ is exactly the condition that we found in the semi–discrete analysis (Theorem 4.1.2) and is found in the continuous analysis Theorem 3.4.3 or [128, p.273].

• We have proved the existence of an absorbing ball in $L^2_{\Delta x}$ without imposing restrictions on the spatial or temporal mesh size.

Having established the existence of the $L^2_{\Delta x}$ absorbing sets we could apply an inverse inequality to find absorbing sets in $H^1_{\Delta x}$. However the bound on the $H^1_{\Delta x}$ norm or the restriction on Δt obtained from this method would be a function of U^0 or Δx. We seek bounds which will hold uniformly as $\Delta x \to 0$.

For the remainder of this section we shall assume $R > 0$ since the dynamics of both the fully discrete system **DI** and continuous system are trivial for $R \leq 0$.

Theorem 4.4.2 (DI3) *Given $U^0 \in L^2_{\Delta x}$ suppose that U^n is a solution of (3.6.12) for all $n > 0$. Then there exists constants $\Delta t_1 > 0$ and $\rho_1 = \rho_1(R) > 0$, ρ_1 independent of Δx, such that $\forall \Delta t < \Delta t_1$ the ball $\mathcal{B}_1(\rho_1)$ is absorbing and positively invariant.*

That is for all $\Delta t < \Delta t_1$, $B \subset \mathcal{B}_0(\rho)$ $\exists$ $n_1(\rho, \rho_1)$:

$$S^n_{\Delta x} B \subset \mathcal{B}_1(\rho_1) \ \forall \ n > n_1$$

where ρ_1 is independent of Δx.

Proof Take the Dirichlet inner–product of (3.6.12) with U^{n+1} :

$$\begin{aligned}\|U^{n+1}\|_1^2 - < U^{n+1}, U^n >_A \ &= \ R\Delta t\|U^{n+1}\|_1^2 - \Delta t(1+i\nu) < M^{-1}AU^{n+1}, U^{n+1} >_A \\ &\quad -\Delta t(1+i\mu) < G(|U^{n+1}|^2)U^{n+1}, U^{n+1} >_A .\end{aligned}$$

Take the real part, and treat the non-linear term as in Theorem (4.1.3):

$$\begin{aligned}&\|U^{n+1}\|_1^2 \\ &= \ \frac{1}{2}\{< U^n, U^{n+1} >_A + < U^{n+1}, U^n >_A\} + R\Delta t\|U^{n+1}\|_1^2 - \Delta t|M^{-1}AU^{n+1}|^2_{L^2_{\Delta x}} \\ &\quad -\Delta t \mathrm{Re}\{(1+i\mu) < G(|U^{n+1}|^2)U^{n+1}, U^{n+1} >_A\} \\ &\leq \ \frac{1}{2}(\|U^n\|_1^2 + \|U^{n+1}\|_1^2) + R\Delta t\|U^{n+1}\|_1^2 - \Delta t|M^{-1}AU^{n+1}|^2_{L^2_{\Delta x}} \\ &\quad -\mathrm{Re}\left\{(1+i\mu)\sum_{j=0}^{J-1}\left[\left|\frac{U^{n+1}_{j+1} - U^{n+1}_j}{\Delta x}\right|^2 (|U^{n+1}_{j+1}|^2 + |U^{n+1}_j|^2)\right.\right. \\ &\quad \left.\left. + U^{n+1}_{j+1}U^{n+1}_j\left(\frac{\overline{U^{n+1}_{j+1}} - \overline{U^{n+1}_j}}{\Delta x}\right)^2\right]\right\} \\ &\leq \ \frac{1}{2}(\|U^n\|_1^2 + \|U^{n+1}\|_1^2) + R\Delta t\|U^{n+1}\|_1^2 - \Delta t|M^{-1}AU^{n+1}|^2_{L^2_{\Delta x}} \\ &\quad +\frac{3}{2}(1+\mu^2)^{1/2}\Delta t\sum_{j=0}^{J-1}\Delta x\left|\frac{U^{n+1}_{j+1} - U^{n+1}_j}{\Delta x}\right|^2 (|U^{n+1}_{j+1}|^2 + |U^{n+1}_j|^2).\end{aligned}$$

Apply Schwarz's inequality to last term and use the periodic boundary conditions to get

$$\begin{aligned}\frac{1}{2}\|U^{n+1}\|_1^2 \ &\leq \ \frac{1}{2}\|U^n\|_1^2 + R\Delta t\|U^{n+1}\|_1^2 - \Delta t|M^{-1}AU^{n+1}|^2_{L^2_{\Delta x}} \\ &\quad +3(1+\mu^2)^{1/2}\Delta t|U^{n+1}|^2_{L^4_{\Delta x}}|DU^{n+1}|^2_{L^4_{\Delta x}}. \qquad (4.4.8)\end{aligned}$$

Hence,

$$(1-2R\Delta t)\|U^{n+1}\|_1^2 \le \|U^n\|_1^2 - 2\Delta t|M^{-1}AU^{n+1}|_{L^2_{\Delta x}}^2 + 6(1+\mu^2)^{1/2}\Delta t|U^{n+1}|_{L^4_{\Delta x}}^2|DU^{n+1}|_{L^4_{\Delta x}}^2$$

which by Lemma 3.7.11 becomes

$$\frac{\|U^{n+1}\|_1^2 - \|U^n\|_1^2}{\Delta t} \le \left(2R + 54(1+\mu^2)|U^{n+1}|_{L^4_{\Delta x}}^4\right)\|U^{n+1}\|_1^2 + |U^{n+1}|_{L^2_{\Delta x}}^2 - |M^{-1}AU^{n+1}|_{L^2_{\Delta x}}^2. \tag{4.4.9}$$

Equation (4.4.9) is of the same form as equation (4.1.13) and looks all set to have the uniform Gronwall lemma applied to it. However the discrete uniform Gronwall lemma 3.4.6 has an extra condition: we require that $\forall\delta \in (0,1)$,

$$\Delta t\left(2R + 54(1+\mu^2)|U^{n+1}|_{L^4_{\Delta x}}^4\right) < 1-\delta.$$

This poses a problem as we do not have any bound on $|U^{n+1}|_{L^4_{\Delta x}}$ (only on the sum from k_0 to k_0+N).

By using the "good" term from the discrete Laplacian (thrown away in the semi–discrete and continuous analysis) we are able to control $L^4_{\Delta x}$ term by an $L^2_{\Delta x}$ term.

First note that if $|U^{n^*}|_{L^2_{\Delta x}}^2 = 0$ for some $n^* \ge 0$ then the desired result is immediate since $\|U^n\|_1^2 = 0 \ \ \forall n \ge n^*$. Without loss of generality we consider $|U^{n+1}|_{L^2_{\Delta x}}^2 \ne 0$.

By Lemma 3.7.10

$$-|M^{-1}AU^{n+1}|_{L^2_{\Delta x}}^2 \le -\frac{\|U^{n+1}\|_1^4}{|U^{n+1}|_{L^2_{\Delta x}}^2},$$

and by the discrete Gagliardo–Nirenberg inequality (Lemma 3.7.9) with $p=4$ and $q=2$ in equation (3.7.26) we find

$$|U^{n+1}|_{L^4_{\Delta x}}^4 \le \left\{|U^{n+1}|_{L^2_{\Delta x}}^4 + |U^{n+1}|_{L^2_{\Delta x}}^3\|U^{n+1}\|_1\right\}.$$

If we now apply the generalized Cauchy–Schwarz inequality we get

$$|U^{n+1}|_{L^4_{\Delta x}}^4 \le \left(1+\frac{1}{\epsilon}\right)|U^{n+1}|_{L^2_{\Delta x}}^4 + \epsilon|U^{n+1}|_{L^2_{\Delta x}}^2\|U^{n+1}\|_1^2. \tag{4.4.10}$$

Hence (4.4.9) becomes :

$$\begin{aligned}
&\frac{\|U^{n+1}\|_1^2 - \|U^n\|_1^2}{\Delta t} \\
\le\ &\left\{2R + 54(1+\mu^2)\left(\left(1+\frac{1}{\epsilon}\right)|U^{n+1}|_{L^2_{\Delta x}}^4 + \epsilon|U^{n+1}|_{L^2_{\Delta x}}^2\|U^{n+1}\|_1^2\right)\right. \\
&\left. -\frac{\|U^{n+1}\|_1^2}{|U^{n+1}|_{L^2_{\Delta x}}^2}\right\}\|U^{n+1}\|_1^2 + |U^{n+1}|_{L^2_{\Delta x}}^2.
\end{aligned} \tag{4.4.11}$$

Now just consider the term in large curly brackets, and let us define ϵ by

$$\epsilon := \frac{1}{54(1+\mu^2)\rho_0^4}.$$

Then

$$\begin{aligned} 2R + 54(1+\mu^2)\left(\left(1+\frac{1}{\epsilon}\right)|U^{n+1}|^4_{L^2_{\Delta x}} + \epsilon|U^{n+1}|^2_{L^2_{\Delta x}}\|U^{n+1}\|_1^2\right) - \frac{\|U^{n+1}\|_1^2}{|U^{n+1}|^2_{L^2_{\Delta x}}} \\ \leq \; 2R + 54(1+\mu^2)(1+54(1+\mu^2)\rho_0^4)|U^{n+1}|^4_{L^2_{\Delta x}} \\ + \|U^{n+1}\|_1^2\left\{\frac{|U^{n+1}|^2_{L^2_{\Delta x}}}{\rho_0^4} - \frac{1}{|U^{n+1}|^2_{L^2_{\Delta x}}}\right\}. \end{aligned}$$

By Theorem 4.4.2 $\forall\ n > n_0,\ |U^n|^2_{L^2_{\Delta x}} \leq \rho_0^2$. Thus $\forall n > n_0$

$$\begin{aligned} 2R + 54(1+\mu^2)\left(\left(1+\frac{1}{\epsilon}\right)|U^{n+1}|^4_{L^2_{\Delta x}} + \epsilon|U^{n+1}|^2_{L^2_{\Delta x}}\|U^{n+1}\|_1^2\right) - \frac{\|U^{n+1}\|_1^2}{|U^{n+1}|^2_{L^2_{\Delta x}}} \\ \leq \; 2R + 54(1+\mu^2)(1+54(1+\mu^2)\rho_0^4)\rho_0^4 + \|U^{n+1}\|_1^2\left\{\frac{1}{\rho_0^2} - \frac{1}{\rho_0^2}\right\} \\ = \; 2R + 54(1+\mu^2)(1+54(1+\mu^2)\rho_0^4)\rho_0^4. \end{aligned} \tag{4.4.12}$$

Substituting (4.4.12) back into the full equation (4.4.11)

$$\frac{\|U^{n+1}\|_1^2 - \|U^n\|_1^2}{\Delta t} \leq \{2R + 54(1+\mu^2)(1+54(1+\mu^2)\rho_0^4)\rho_0^4\}\,\|U^{n+1}\|_1^2 + |U^{n+1}|^2_{L^2_{\Delta x}}.$$

Define Δt_1 such that for fixed $\delta\ \in (0,1)$

$$\Delta t_1 := \frac{1-\delta}{2R + 54(1+\mu^2)(1+54(1+\mu^2)\rho_0^4)\rho_0^4}. \tag{4.4.13}$$

We may now apply the discrete uniform Gronwall Lemma 3.7.18 with constants a_1, a_2 and a_3 found from equation (4.4.7) with $r = N\Delta t$, so that $\forall\ k_0 > n_0$

$$\begin{aligned} \sum_{n=k_0}^{N+k_0}\Delta t\left(2R + 54(1+\mu^2)(1+\frac{2}{\epsilon})|U^{n+1}|_{L^2_{\Delta x}}\right) &\leq \left(2R + 54(1+\mu^2)(1+\frac{2}{\epsilon^2})\rho_0^4\right) r \\ &:= a_1 \end{aligned}$$

$$\sum_{n=k_0}^{N+k_0} 2|U^{n+1}|^2_{L^2_{\Delta x}}\Delta t \leq \left(4rR^2 + \rho_0^2\right)/2R := a_2$$

$$\sum_{n=k_0}^{N+k_0}\|U^{n+1}\|_1^2\Delta t \leq \left(4rR^2 + \rho_0^2\right)/2 := a_3$$

and noting that by (4.4.13)

$$\Delta t \; < \; \frac{1-\delta}{2R+54(1+\mu^2)(1+54(1+\mu^2)\rho_0^4)\rho_0^4}.$$

Thus

$$||U^{k_0+1}||_1^2 \leq \left(a_2 + \frac{a_3}{r}\right)\exp(\frac{a_1}{\delta}) \qquad \forall \;\; k_0 \geq n_0 + N.$$

Hence we have a uniform bound on the $H^1_{\Delta x}$ semi–norm which is independent of the initial condition U^0, the spatial step size Δx and the temporal step size Δt provided $\Delta t < \Delta t_1$ where Δt_1 satisfies relation (4.4.13).

Therefore we have absorbing balls $\mathcal{B}_1(\rho_1)$ in $H^1_{\Delta x}$ with $n_1 = k_0$, and

$$\rho_1^2 = \rho_0^2 + \left(a_2 + \frac{a_3}{r}\right)\exp(\frac{a_1}{\delta}),$$

ρ_1 independent of Δx. □

▷ The previous theorem fulfills our goal of a bound on Δt independent of the initial data and spatial step size. This was achieved by using the 'Laplacian' term $|M^{-1}AU^{n+1}|^2_{L^2_{\Delta x}}$. We also note that there is no restriction on the spatial step size Δx and that the only restriction on Δt arises from the application of the discrete Gronwall lemma 3.7.18.

The following theorem presents two other approaches we devised for handling the extra condition from the discrete uniform Gronwall lemma 3.4.6. For small and restrictive values of R it is possible to restrict Δt independently Δx and control the $L^4_{\Delta x}$ norm. For larger values of R we apply an inverse inequality. Both approaches yield bounds on the semi–norm which are independent of Δx.

Theorem 4.4.3 *Suppose given $U^0 \in L^2_{\Delta x}$ that U^{n+1} exists $\forall n > 0$ and consider the following restrictions on $\Delta t, R$ and μ:*

1. *For $R \in (0, \frac{1}{18})$, $\mu^2 < 1 - \frac{1}{18R}$*

$$\Delta t < \frac{1 - 9(1+\mu^2)\rho_0^2}{2R + 54(1+\mu^2)R\rho_0^2},$$

where ρ_0 is as in Theorem 4.4.1, so that Δt is independent of $||U^0||_1$ and Δx;

2. *For $R, \mu \in I\!R$,*

$$\Delta t < \frac{1-\delta}{2R + 54(1+\mu^2)\rho_0^4\Delta x^{-1}}$$

where ρ_0 is as in Theorem 4.4.1, so that Δt is independent of $\|U^0\|_1$ but dependent on Δx.

Then for restrictions 1. and 2. there exists $a_1, a_2, a_3 \in I\!R^+$ such that for $r = (N+1)\Delta t$ and $\forall\ k_0 > n_0 + N,\ \delta \in (0,1)$

$$\|U^{k_0+1}\|_1 \leq \left(\frac{a_3}{r} + a_2\right) \exp\left(\frac{a_1}{\delta}\right).$$

Proof

Suppose equation (4.4.9) is established in exactly the same manner as in Theorem 4.4.2. We seek to bound $|U^{n+1}|_{L^4_{\Delta x}}$ directly so we may apply the uniform Gronwall lemma 3.4.6 directly to (4.4.9).

1. For $R \in (0, \frac{1}{18})$ and $\mu^2 < 1 - \frac{1}{18R}$.

   We use the scheme **DI** (3.6.12) to find a bound the $L^4_{\Delta x}$ norm. Rearrange equation (4.4.2) to get

$$\Delta t|U^{n+1}|^4_{L^4_{\Delta x}} \leq \frac{1}{2}(|U^n|^2_{L^2_{\Delta x}} - |U^{n+1}|^2_{L^2_{\Delta x}}) + R\Delta t|U^{n+1}|^2_{L^2_{\Delta x}}$$

   and recall that by Theorem 4.4.1 we are able to bound the $L^2_{\Delta x}$ norm so that

$$\Delta t|U^{n+1}|^4_{L^4_{\Delta x}} \leq \frac{1}{2}\rho_0^2 + R\Delta t\rho_0^2\ \ \forall\ n > n_0.$$

   Then,

$$\Delta t 2R + 54(1+\mu^2)\Delta t|U^{n+1}|^4_{L^4_{\Delta x}} \leq \Delta t 2R + 54(1+\mu^2)(\frac{1}{2}\rho_0^2 + R\Delta t\rho_0^2)$$

   and

$$\begin{aligned}
\Delta t 2R + 54(1+\mu^2)(\frac{1}{2}\rho_0^2 + R\Delta t\rho_0^2) &\leq 1-\delta \\
\Rightarrow \Delta t(2R + 54(1+\mu^2)R\rho_0^2) &< 1 - 9(1+\mu^2)\rho_0^2 \\
\Rightarrow \Delta t &< \frac{1 - 9(1+\mu^2)\rho_0^2}{2R + 54(1+\mu^2)R\rho_0^2}.
\end{aligned}$$

   Therefore we require that

$$1 - 9(1+\mu^2)\rho_0^2 > 0.$$

   Recall that $\rho_0 > 2R$, so we require exactly that

$$0 < R < \frac{1}{18} \text{ and } \mu^2 < 1 - \frac{1}{18R}.$$

2. For $R, \mu \in \mathbb{R}$

Here we use the inverse inequality proved in Lemma 3.7.12 in equation (3.7.24) to get

$$|U^{n+1}|^4_{L^4_{\Delta x}} \leq \Delta x^{-1} |U^{n+1}|^4_{L^2_{\Delta x}}. \tag{4.4.14}$$

Thus for $n \geq n_0$

$$|U^{n+1}|^2_4 \leq \Delta x^{-1} \rho_0^4$$

and we choose Δt so that

$$\Delta t < \frac{1-\delta}{2R + 54(1+\mu^2)\rho_0^4 \Delta x^{-1}}.$$

We now show the other conditions hold for the discrete uniform Gronwall inequality. To this effect we use the bound from equation (4.4.7) to find the constants a_1, a_2 and a_3.

$$\sum_{n=k_0}^{N+k_0} (2R + 54(1+\mu^2)|U^{n+1}|^4_{L^4_{\Delta x}})\Delta t \leq 12(1+\mu^2)\left(4rR^2 + \rho_0^2\right) := a_1,$$

$$\sum_{n=k_0}^{N+k_0} 2|U^{n+1}|^2_{L^2_{\Delta x}} \Delta t \leq \left(4rR^2 + \rho_0^2\right)/2R := a_2,$$

$$\sum_{n=k_0}^{N+k_0} \|U^{n+1}\|_1^2 \Delta t \leq \left(4rR^2 + \rho_0^2\right)/2 := a_3.$$

Thus using the discrete uniform Gronwall Lemma 3.4.6

$$\|U^{k_0+1}\|_1^2 \leq \exp\left(\frac{a_1}{\delta}\right)(a_2 + a_3) \;\; \forall k_0 \geq n_0 + N$$

and the Theorem is proved.□

Notes

• In 1 dimension we could avoid the need to bound the $L^4_{\Delta x}$ norm by discretizing the argument suggested by Dr. E. Süli, (see section 3.1). We note that the method employed in Theorem 4.4.2 which exploits the Laplacian term has the advantage that it extends to two dimensions.

• The $H^1_{\Delta x}$ bounds established in Theorem 4.1.5 could easily be derived in this fully discrete context if desired.

Thus far we have been rather coy about the existence of a continuous semi–group for (3.6.12) or equivalently the existence and uniqueness **DI1** of a solution to our discrete system which is continuously dependent on the initial condition.

The following theorem which is stated in the real case in [28, p.58] or [47] gives an approach for proving existence.

Theorem 4.4.4 *Let B be a closed ball in $\mathbb{C}^n$. Suppose $\Phi : B \to \mathbb{C}^n$ is continuous and satisfies*

$$Re\ \{< \Phi(v), v >\} < 0 \qquad \forall v \in \partial B.$$

Then there exists $v \in B$ such that $\Phi(v) = 0$.

Proof Let the radius of B be ρ and assume that

$$\Phi(v) \neq 0 \ \ \forall v \in B.$$

Then $v \to \rho\Phi(v)/|\Phi(v)|$ is a map from B into ∂B. As a continuous map from B to B by Brouwer's fixed point theorem it has a fixed point, v_0, say. So,

$$v_0 = \rho\Phi(v_0)/|\Phi(v_0)|$$

and furthermore $v_0 \in \partial B$. This yields a contradiction since $< v_0, v_0 > \geq 0$ and

$$\mathrm{Re}(< v_0, v_0 >) = \mathrm{Re}\left(\rho\frac{< \Phi(v_0), v_0 >}{|\Phi(v_0)|}\right) < 0. \qquad \square$$

The following Theorem gives us the existence of a solution for the system (3.6.12).

Theorem 4.4.5 (Existence) *There exists a solution U of the equation*

$$(1 + i\nu)MU - R\Delta tU - \Delta t \qquad\qquad {}^{-1}AU - \Delta t(1 + i\mu)G(|U|^2)U = V$$

for all $\Delta t \geq 0$ and all $V \in \mathbb{C}^J_{\mathrm{per}}$.

Proof Let V be given. Our task is to show that we can find U. To do this we apply Theorem 4.4.4. Let B be the ball of radius $\sqrt{2}\rho$ in $\mathbb{C}^J_{\mathrm{per}}$, where

$$\rho^2 = \frac{1}{1 + 2R\Delta t}(|V|^2_{L^2_{\Delta x}} + 4\Delta tR^2)$$

and define $\Phi : B \to \mathbb{C}^J_{\mathrm{per}}$ by

$$-\Phi(U) = 2(U - V) - 2R\Delta tU - \Delta t(1 + i\nu)M^{-1}AU - \Delta t(1 + i\mu)G(|U|^2)U.$$

Take the $L^2_{\Delta x}$ inner–product of $\Phi(U)$ with U to get

$$\begin{aligned}-<\Phi(U),U> &= 2|U|^2_{L^2_{\Delta x}} - 2<V,U> -2\Delta t\Big(R|U|^2_{L^2_{\Delta x}} - (1+i\mu)<M^{-1}AU,U> \\ &\quad -(1+i\mu)<G(|U|^2)U,U>\Big),\end{aligned}$$

and take the real part to find

$$\begin{aligned}-\mathrm{Re}\,\langle\Phi(U),U\rangle &= 2|U|^2_{L^2_{\Delta x}} - (<V,U>+<U,V>) \\ &\quad - 2\Delta t(R|U|^2_{L^2_{\Delta x}} - \|U\|_1^2 - |U|^4_{L^4_{\Delta x}}).\end{aligned}$$

Now use (4.1.5) to bound half the $L^4_{\Delta x}$ norm

$$\begin{aligned}-\mathrm{Re}\{<\Phi(U),U>\} &\geq 2|U|^2_{L^2_{\Delta x}} - |U|^2_{L^2_{\Delta x}} - |V|^2_{L^2_{\Delta x}} + 2R^2\Delta t\|U\|_1^2 + 2R\Delta t|U|^2_{L^2_{\Delta x}} \\ &\quad - 2R^2\Delta t + \Delta t|U|^4_{L^4_{\Delta x}} \\ &\geq (1+2R\Delta t)|U|^2_{L^2_{\Delta x}} - (|V|^2_{L^2_{\Delta x}} + 4\Delta t R^2).\end{aligned}$$

Now for $U \in \partial B$ we see

$$-\mathrm{Re}<\Phi(U),U> \geq (1+2R\Delta t)(2\rho_1^2 - \rho_1^2) > 0.$$

Therefore, by Theorem 4.4.4, we have a solution. □

Theorem 4.4.6 (Uniqueness) *Given $U^0 \in L^2_{\Delta x}$ suppose that Δt satisfies*

$$\Delta t < \Delta t_2 := \frac{1}{2(R+2(1+\mu^2)^{1/2})\rho_0^2}, \tag{4.4.15}$$

where ρ_0 is the radius of the $L^2_{\Delta x}$ absorbing ball of Theorem 4.4.1.

Then there exists a unique solution U^n to (3.6.12) $\forall n \geq n_0$ and U^n is continuously dependent on the initial data U^0.

Proof First we examine uniqueness.

Suppose for a given U^n we had two solutions U^{n+1}, V^{n+1} satisfying,

$$U^{n+1} = U^n - \Delta t\left\{RU^{n+1} - (1+i\nu)M^{-1}AU^{n+1} - (1+i\mu)G(|U^{n+1}|^2)U^{n+1}\right\}$$

and

$$V^{n+1} = U^n - \Delta t\left\{RV^{n+1} - (1+i\nu)M^{-1}AV^{n+1} - (1+i\mu)G(|V^{n+1})V^{n+1}\right\}.$$

Then subtracting, taking the inner–product with $U^{n+1}-V^{n+1}$, and taking the real part gives

$$\begin{aligned}|U^{n+1}-V^{n+1}|^2_{L^2_{\Delta x}} &= R\Delta t|U^{n+1}-V^{n+1}|^2_{L^2_{\Delta x}} - \Delta t\|M^{-1}A(U^{n+1}-V^{n+1})\|_1\\ &\quad -\mathrm{Re}\left\{\Delta t(1+i\mu) < G(|U^{n+1}|^2)U^{n+1} - G(|V^{n+1})V^{n+1}, U^{n+1}-V^{n+1} >\right\}.\end{aligned}$$

Let us just consider the non–linear term and apply lemma 3.7.20 to it and take the modulus to get

$$\begin{aligned}&-\mathrm{Re}\left\{\Delta t(1+i\mu) < G(|U^{n+1}|^2)U^{n+1} - G(|V^{n+1})V^{n+1}, U^{n+1}-V^{n+1} >\right\}\\ &\leq \Delta t(1+\mu^2)^{1/2}\left(|U^{n+1}|^2_{L^2_{\Delta x}} + |V^{n+1}|^2_{L^2_{\Delta x}}\right)|U^{n+1}-V^{n+1}|^2_{L^2_{\Delta x}}. \qquad (4.4.16)\end{aligned}$$

So,

$$\begin{aligned}|U^{n+1}-V^{n+1}|^2_{L^2_{\Delta x}} &\leq R\Delta t|U^{n+1}-V^{n+1}|^2_{L^2_{\Delta x}} - \Delta t\|M^{-1}A(U^{n+1}-V^{n+1})\|_1\\ &\quad + \Delta t(1+\mu^2)^{1/2}\left(|U^{n+1}|^2_{L^2_{\Delta x}} + |V^{n+1}|^2_{L^2_{\Delta x}}\right)|U^{n+1}-V^{n+1}|^2_{L^2_{\Delta x}}.\end{aligned}$$

Hence for $n > n_0$, we find

$$1 \geq \Delta t R + 2\Delta t(1+\mu^2)^{1/2}\rho_0^2,$$

and provided the restriction (4.4.15) holds, we have a unique solution .

We establish continuity with respect to initial data by proving Lipschitz continuity. Let U^{n+1}, V^{n+1} be two solutions to 3.6.12 at step $n+1$ with initial conditions U^0 and V^0 respectively. Subtracting the equations for U^{n+1} and V^{n+1}, taking the $L^2_{\Delta x}$ innerproduct with $U^{n+1}-V^{n+1}$, taking the real part and using (4.4.16) we get

$$\begin{aligned}&\frac{|U^{n+1}-V^{n+1}|^2_{L^2_{\Delta x}} - |U^n-V^n|^2_{L^2_{\Delta x}}}{2\Delta t}\\ &\leq R|U^{n+1}-V^{n+1}|^2_{L^2_{\Delta x}} - \|M^{-1}A(U^{n+1}-V^{n+1})\|\\ &\quad + (1+\mu^2)^{1/2}\left(|U^{n+1}|^2_{L^2_{\Delta x}} + |V^{n+1}|^2_{L^2_{\Delta x}}\right)|U^{n+1}-V^{n+1}|^2_{L^2_{\Delta x}}\\ &\leq R|U^{n+1}-V^{n+1}|^2_{L^2_{\Delta x}} + (1+\mu^2)^{1/2}\rho_0^2|U^{n+1}-V^{n+1}|^2_{L^2_{\Delta x}}.\end{aligned}$$

Thus we find that

$$|U^{n+1}-V^{n+1}|^2_{L^2_{\Delta x}} \leq \frac{1}{1-2R\Delta t - 2\Delta t(1+\mu^2)^{1/2}\rho_0^2}|U^n-V^n|^2_{L^2_{\Delta x}},$$

and by the choice of Δt the Lemma is proved. □

Theorem 4.4.7 (DI4) *Suppose that Δt is restricted so that*

$$\Delta t < \Delta t_3 := \min\{1, \Delta t_1, \Delta t_2\} \tag{4.4.17}$$

where Δt_1, Δt_2 are given by (4.4.13) *and* (4.4.15). *Then the dynamical system given by the fully implicit discrete Ginzburg–Landau equation* (3.6.12) *possesses a global attractor $\mathcal{A}_{\Delta x}$,*

$$\mathcal{A}_{\Delta x} = \omega\left(\mathcal{B}_1(\rho_1)\right)$$

which is bounded independently of Δx in $H^1_{\Delta x}$.

The attractor $\mathcal{A}_{\Delta x}$ is compact, connected and maximal in $L^2_{\Delta x}$. Furthermore $\mathcal{A}_{\Delta x}$ is given by

$$\mathcal{A}_{\Delta x} = \omega(\mathcal{B}_1(\rho_1))$$

and is bounded in $H^1_{\Delta x}$ independently of Δx.

Proof All that is required is to show the conditions for Theorem 1.2.1 hold. Theorems 4.4.3 and 4.4.4 give us all we require. □

4.4.2 The Explicit–Implicit Scheme DEI

In this section we consider the scheme **DEI** (3.6.16) and prove existence of an absorbing ball in $L^2_{\Delta x}$. We recall that for the fully implicit scheme (3.6.12) we were able to obtain results with restrictions on Δt independent of Δx. Regrettably for the mixed scheme **DEI** (3.6.16) we were unable to prove such a result and instead have restrictions on Δt in terms of Δx. From numerical evidence (see section 4.5.3) this is more likely to be a failing of the analysis than a true reflection of the properties of the scheme.

We commence with some lemmas which establish bounds that will be used to control the non–linear term in the proofs of **DEI2**.

Lemma 4.4.1 *Given any vector $U = (U_0, \ldots, U_{J-1})^T \in \mathbb{C}^J_{\text{per}}$, with $\delta_+, \delta_-, \delta^2$ given by* (3.6.1) *and* (3.6.2) *then*

$$Re\left\{\delta^2 U_j \overline{U_j}\right\} = \frac{1}{2}\delta^2|U_j|^2 \frac{1}{2}|\delta_+ U_j|^2 - \frac{1}{2}|\delta_- U_j|^2.$$

Proof For the purposes of the proof we shall multiply through by Δx^2 to get

$$\begin{aligned}
&\mathrm{Re}\{\Delta x^2\delta^2(U_j)\overline{U_j}\} \\
&= \frac{1}{2}\left\{U_{j+1}\overline{U_j} - 2|U_j|^2 + U_{j-1}\overline{U_j} + U_j\overline{U}_{j+1} + U_j\overline{U}_{j-1} - 2|U_j|^2\right\} \\
&= \frac{1}{2}\left\{|U_{j+1}|^2 - 2|U_j|^2 + |U_{j-1}|^2 - |U_{j+1}|^2 - 2|U_j|^2 - |U_{j-1}|^2 + U_{j+1}\overline{U_j}\right. \\
&\quad \left. + U_j\overline{U}_{j+1} + U_{j-1}\overline{U}_j + U_j\overline{U}_{j-1}\right\} \\
&= \frac{1}{2}\Delta x^2\delta^2|U_j|^2 + \frac{1}{2}\left\{-|U_{j+1}|^2 - |U_j|^2 + U_{j+1}\overline{U_j} + U_j\overline{U}_{j+1} - |U_{j-1}|^2 - |U_j|^2\right. \\
&\quad \left. + U_{j-1}\overline{U_j} + U_j\overline{U}_{j-1}\right\} \\
&= \frac{1}{2}\Delta x^2\delta^2|U_j|^2 - \frac{1}{2}\left\{|U_{j+1} - U_j|^2 + |U_{j-1} - U_j|^2\right\},
\end{aligned}$$

and hence we have the result.□

Lemma 4.4.2 *Suppose that given $U^0 \in L^2_{\Delta x}$ there exists a solution U^n of* **DEI** (3.6.16) *for all $n > 0$. Then,*

$$|U_j^{n+1}|^2 - |U_j^n|^2 \le 2R\Delta t|U_j^{n+1}|^2 + 2(1+\nu^2)^{1/2}\Delta t\left|(\delta^2 U_j^{n+1})\overline{U}_j^{n+1}\right|. \qquad (4.4.18)$$

Proof Write the scheme (3.6.16) in component form

$$U_j^{n+1} = U_j^n + R\Delta t U_j^{n+1} + \Delta t(1+i\nu)\delta^2 U_j^{n+1} - \Delta t(1+i\mu)|U_j^n|^2 U_j^{n+1}$$

multiply by $\overline{U}_j^{n+1}$ and take the real part to get

$$\begin{aligned}
|U_j^{n+1}|^2 &= \frac{1}{2}(U_j^n\overline{U}_j^{n+1} + U_j^{n+1}\overline{U}_j^n) + R\Delta t|U_j^{n+1}|^2 + \Delta t\mathrm{Re}\left\{(1+i\nu)(\delta^2 U_j^{n+1})\overline{U}_j^{n+1}\right\} \\
&\quad -\Delta t|U_j^n|^2|U_j^{n+1}|^2.
\end{aligned}$$

Completing the square and throwing away the term from the non–linearity yields

$$|U_j^{n+1}|^2 - |U_j^n|^2 \le 2R\Delta t|U_j^{n+1}|^2 + 2(1+\nu^2)^{1/2}\Delta t\left|(\delta^2 U_j^{n+1})\overline{U}_j^{n+1}\right|.\square$$

Lemma 4.4.3 *Suppose that given $U^0 \in L^2_{\Delta x}$ there exists a solution U^n of* (3.6.16) *for all $n > 0$. Then we have the following two inequalities*

$$\sum_{j=0}^{J-1}\Delta x\left|\left(\delta^2 U_j^{n+1}\right)\overline{U}_j^{n+1}\right||U_j^{n+1}|^2 \le \frac{1}{2}|U^{n+1}|^2_{L^\infty_{\Delta x}}|M^{-1}AU^{n+1}|^2_{L^2_{\Delta x}} + \frac{1}{2}|U^{n+1}|^4_{L^4_{\Delta x}} \qquad (4.4.19)$$

and,

$$\sum_{j=0}^{J-1}\Delta x\left|\left(\delta^2 U_j^{n+1}\right)\overline{U_j^{n+1}}\right||U_j^{n+1}|^2 \le \frac{4}{\Delta x^2}|U^{n+1}|^4_{L^4_{\Delta x}}. \qquad (4.4.20)$$

Proof To find (4.4.19) complete the square and use the periodic boundary conditions

$$\sum_{j=0}^{J-1} \Delta x \left| \left(\delta^2 U_j^{n+1}\right) \overline{U_j^{n+1}} \right| |U_j^{n+1}|^2 \leq \frac{1}{2} \sum_{j=0}^{J-1} \Delta x \left| \left(\delta^2 U_j^{n+1}\right) \overline{U_j^{n+1}} \right|^2 + \frac{1}{2} \sum_{j=0}^{J-1} \Delta x |U_j^{n+1}|^4$$

and (4.4.19) is immediate.

The second inequality (4.4.20) requires a little more work.

$$\begin{aligned}
&\sum_{j=0}^{J-1} \Delta x \left| \left(\delta^2 U_j^{n+1}\right) \overline{U_j^{n+1}} \right| |U_j^{n+1}|^2 \\
&= \sum_{j=0}^{J-1} \Delta x \left| \frac{(U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1})}{\Delta x^2} \overline{U_j^{n+1}} \right| |U_j^{n+1}|^2 \\
&\leq \sum_{j=0}^{J-1} \Delta x \frac{(|U_{j+1}^{n+1}||U_j^{n+1}| + 2|U_j^{n+1}|^2 + |U_{j-1}^{n+1}||U_j^{n+1}|)}{\Delta x^2} |U_j^{n+1}|^2.
\end{aligned}$$

Complete the square twice to get:

$$\begin{aligned}
&\sum_{j=0}^{J-1} \Delta x \left| \delta^2 U_j^{n+1} \overline{U_j^{n+1}} \right| |U_j^{n+1}|^2 \\
&\leq \frac{1}{\Delta x^2} \sum_{j=0}^{J-1} \Delta x \frac{1}{2} (|U_{j+1}^{n+1}|^2 + |U_j^{n+1}|^2 + 4|U_{j+1}^{n+1}|^2 + |U_{j-1}^{n+1}|^2 + |U_j^{n+1}|^2) |U_j^{n+1}|^2 \\
&\leq \frac{1}{\Delta x^2} \sum_{j=0}^{J-1} \Delta x \frac{1}{4} (|U_{j+1}^{n+1}|^4 + |U_j^{n+1}|^4 + 2|U_j^{n+1}|^4 + 2|U_j^{n+1}|^2 \\
&+ |U_{j-1}^{n+1}|^4 + |U_j^{n+1}|^4 + |U_{j-1}^{n+1}|^4 + |U_j^{n+1}|^4);
\end{aligned}$$

and use the periodic boundary conditions

$$\begin{aligned}
\sum_{j=0}^{J-1} \Delta x | \left(\delta^2 U_j^{n+1}\right) \overline{U_j^{n+1}} | |U_j^{n+1}|^2 &\leq \frac{4}{\Delta x^2} \sum_{j=0}^{J-1} \Delta x |U_j^{n+1}|^4 \\
&= \frac{4}{\Delta x^2} |U^{n+1}|_{L^4_{\Delta x}}^4
\end{aligned}$$

to get (4.4.20). □

We are now in a position to establish **DEI2** : the existence of an absorbing ball $\mathcal{B}_0(\rho_0)$ where ρ_0 is independent of Δx.

Theorem 4.4.8 (DEI2) *Suppose that given $U^0 \in L^2_{\Delta x}$ we have a solution U^n of* (3.6.16) $\forall n > 0$. *Then there exists a constant $\rho_0 = \rho_0(R) > 0$, independent of Δx, such that $\mathcal{B}_0(\rho_0)$ is absorbing and positively invariant provided Δt satisfies*

$$\Delta t < \Delta t_1 := \left(\frac{\Delta x^2}{4R\Delta x^2 + 16(1+\nu^2)^{1/2}} \right). \tag{4.4.21}$$

That is $\forall\ \Delta t < \Delta t_1$, *for* $B \subset \mathcal{B}_0(\rho)\ \exists\ n_1 = n_1(\rho, \rho_0)$ *s.t.*$\forall n > n_1$,

$$S^n_{\Delta x} B \subset \mathcal{B}_0(\rho_0),$$

where ρ_0 *is independent of* Δx.

Proof For $R \leq 0$ the proof is exactly like that of Theorem 4.4.2, and we may take ρ_0 to be any positive constant.

For $R > 0$ we take the $L^2_{\Delta x}$ inner product of (3.6.16) with U^{n+1}, take the real part, and complete the square to find

$$\frac{|U^{n+1}|^2_{L^2_{\Delta x}} - |U^n|^2_{L^2_{\Delta x}}}{2\Delta t} \leq R|U^{n+1}|^2_{L^2_{\Delta x}} - \|U^{n+1}\|_1^2 + \sum_{j=0}^{J-1} \Delta x |U_j^n|^2 |U_j^{n+1}|^2. \tag{4.4.22}$$

Now add in and subtract out $\Delta t|U^{n+1}|^4_{L^4_{\Delta x}}$ on the right–hand side. This converts (4.4.22) to the fully implicit case (4.4.2) with an error term. We successfully dealt with (4.4.2) so our main concern will be the error term.

$$\begin{aligned}\frac{|U^{n+1}|^2_{L^2_{\Delta x}} - |U^n|^2_{L^2_{\Delta x}}}{2\Delta t} \leq\ & R|U^{n+1}|^2_{L^2_{\Delta x}} - \|U^{n+1}\|_1^2 - |U^{n+1}|^4_{L^4_{\Delta x}} \\ & + \sum_{j=0}^{J-1} \Delta x (|U_j^{n+1}|^2 - |U_j^n|^2)|U_j^{n+1}|^2.\end{aligned}$$

Use the point–wise bound that we found in Lemma 4.4.2 on the error term

$$\begin{aligned}&\frac{|U^{n+1}|^2_{L^2_{\Delta x}} - |U^n|^2_{L^2_{\Delta x}}}{2\Delta t} \\ &\leq\ R|U^{n+1}|^2_{L^2_{\Delta x}} - |U^{n+1}|^4_{L^4_{\Delta x}} - \|U^{n+1}\|_1^2 \\ &\quad + \sum_{j=0}^{J-1} \Delta x \left(2R\Delta t|U_j^{n+1}|^2 + 2(1+\nu^2)^{1/2}\Delta t \left|\left(\delta^2 U_j^{n+1}\right) \overline{U}_j^{n+1}\right|\right) |U_j^{n+1}|^2 \\ &\leq\ R|U^{n+1}|^2_{L^2_{\Delta x}} - |U^{n+1}|^4_{L^4_{\Delta x}} - \|U^{n+1}\|_1^2 + 2R\Delta t \sum_{j=0}^{J-1} \Delta x |U_j^{n+1}|^4 \\ &\quad + 2(1+\nu^2)^{1/2}\Delta t \sum_{j=0}^{J-1} \Delta x \left|\delta^2 U_j^{n+1} \overline{U}_j^{n+1}\right| |U_j^{n+1}|^2.\end{aligned}$$

By Lemma 4.4.3 inequality (4.4.20)

$$\begin{aligned}\frac{|U^{n+1}|^2_{L^2_{\Delta x}} - |U^n|^2_{L^2_{\Delta x}}}{2\Delta t} \leq\ & R|U^{n+1}|^2_{L^2_{\Delta x}} - \|U^{n+1}\|_1^2 - \frac{1}{2}|U^{n+1}|^4_{L^4_{\Delta x}} - \frac{1}{2}|U^{n+1}|^4_{L^4_{\Delta x}} \\ & + \left(2R\Delta t + 8(1+\nu^2)^{1/2}\frac{\Delta t}{\Delta x^2}\right) |U^{n+1}|^4_{L^4_{\Delta x}}.\end{aligned}$$

Hence

$$
\begin{aligned}
|U^{n+1}|_{L^2_{\Delta x}}^2 \le\ & |U^n|_{L^2_{\Delta x}}^2 + 2R\Delta t|U^{n+1}|_{L^2_{\Delta x}}^2 - \Delta t|U^{n+1}|_{L^4_{\Delta x}}^4 - 2\Delta t\|U^{n+1}\|_1^2 \\
& + \Delta t\left\{4R\Delta t + 16(1+\nu^2)^{1/2}\frac{\Delta t}{\Delta x^2} - 1\right\}|U^{n+1}|_{L^4_{\Delta x}}^4,
\end{aligned}
$$

and we can control the error term provided the restriction (4.4.21) holds. Thus

$$
|U^{n+1}|_{L^2_{\Delta x}}^2 \le |U^n|_{L^2_{\Delta x}}^2 - 2\Delta t\|U^{n+1}\|_1^2 + 2R\Delta t|U^{n+1}|_{L^2_{\Delta x}}^2 - \Delta t|U^{n+1}|_{L^4_{\Delta x}}. \qquad (4.4.23)
$$

We are now in exactly the same position as in Theorem 4.4.1: inequality (4.4.3), and the existence of an absorbing set in $L^2_{\Delta x}$ follows by the same analysis. □

Notes

- The summation bounds of inequality (4.4.7) also follows for this explicit – implicit scheme in exactly the same way.
- Although we sought an absorbing ball in the discrete space $H^1_{\Delta x}$ using the energy type arguments of the previous theorems we were unsuccesful. We recall hoever that due to the finite dimensionality of the problem for any fixed Δx we may find such a bound on the $H^1_{\Delta x}$ norm - however this fails to converge as $\Delta x \to 0$.

Theorem 4.4.9 (DEI4) *Suppose that $\Delta t < \Delta t_2$ where Δt_2 is given by* (??) *and that the conditions on $\frac{\Delta t}{\Delta x}$* (4.4.21) *and* (??) *hold. Further assume that the scheme* **DEI** *forms a dynamical system. Then the dynamical system given by the explicit implicit discrete Ginzburg-Landau equation* (3.6.16) *possesses a global attractor $\mathcal{A}_{\Delta x}$,*

$$
\mathcal{A}_{\Delta x} = \omega\left(\mathcal{B}_1(\rho_0)\right).
$$

Proof All that is required is to verify the conditions of Theorem 1.2.1 hold, and Theorem4.4.8 gives us all we require using the finite dimensionality of the problem. □

Note In section 4.5 we present numerical evidence that Δt may be taken to be independent of the spatial mesh Δx.

4.5 Numerical Results

In this Section we present some numerical results for the schemes **DFE** (3.6.14), **DE** (3.6.15), **DEI** (3.6.16) and **DI** (3.6.12). Each scheme was investigated in a methodical manner against the continuous results of Section 3.4 and our results of this chapter. For these computer simulations the value of μ was fixed at $\mu = -\sqrt{3}$ and for various values of ν we varied the bifurcation parameter R. First we examined for various initial conditions the parameter regime $R \leq 0$, a regime for which we know that all initial conditions for the continuous problem tend to the trivial solution $U \equiv 0$ (see section 3.4). We then examined the regime $R > 0$ to test numerically the existence of an absorbing set independent of initial conditions and also the existence and stability of the rotating wave solutions.

4.5.1 The Schemes DFE and DE

The fully explicit scheme **DFE** (3.6.14) and the scheme **DE** (3.6.15) with a fully explicit non–linear term both displayed dependence on initial conditions for values of $R < 0$. As stated above the behaviour of the continuous equation is independent of the initial data. For large initial conditions (for example $|U^0|_{L^2_{\Delta x}} \geq 100$) we found that the schemes would fail to converge, whereas for smaller initial data the correct behaviour was observed. This is illustrated for the scheme **DFE** (3.6.14) in Figures 4.1 and 4.2 with $R = -10$, $\mu = -\sqrt{3}$, $\nu = \sqrt{3}$, $\Delta x = 1/64$ and $\Delta t = 0.0001$. In Figure 4.1 the norm of the initial data was $|U^0|^2_{L^2_{\Delta x}} = 20000$ and the scheme fails to converge where as in Figure 4.2 the initial condition has norm $|U^0|^2_{L^2_{\Delta x}} = 2000$ and we have the expected exponential convergence to the trivial solution $U \equiv 0$.

For the scheme **DE** (3.6.15) the same phenomena of dependence on initial conditions is observed and this is illustrated in Figure 4.3 for $R = -10, \mu = -\sqrt{3}, \nu = \sqrt{3}, \Delta x = 1/64$ and $\Delta t = 0.0001$. The figure shows divergence of the scheme for large initial data and convergence of the scheme for smaller initial data.

These numerical results indicate that we could not hope to repeat the analysis of Section 4.4 for the schemes **DFE** and **DE** and find an absorbing ball $\mathcal{B}_0(\rho)$ with ρ independent of initial data U^0. In this sense these schemes are not good approximations to the Ginzburg–Landau equation.

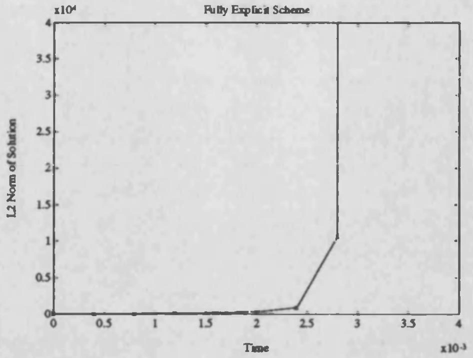

Figure 4.1: Failure to converge with large initial data for the scheme **DFE**: $R = -10$, $\nu = \sqrt{3}$, $\Delta x = 1/64$, $\Delta t = 0.0001$.

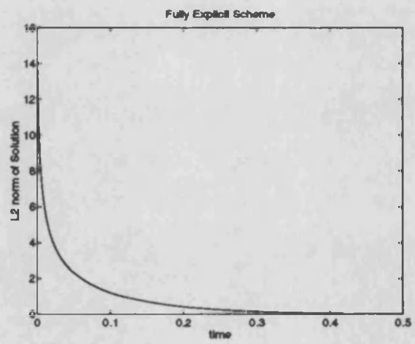

Figure 4.2: Convergence with smaller initial data for the scheme **DFE**: $R = -10$, $\nu = \sqrt{3}$, $\Delta x = 1/64$, $\Delta t = 0.0001$.

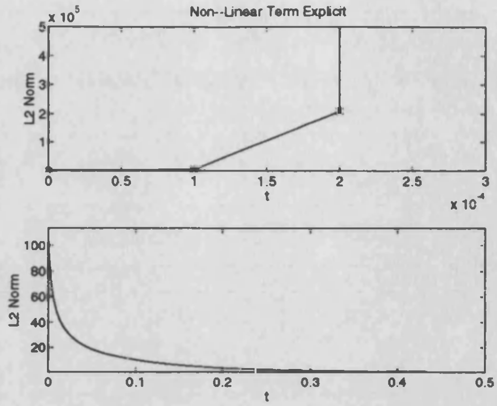

Figure 4.3: Dependence on initial data for the scheme **DE**: $R = -10$, $\nu = \sqrt{3}$, $\Delta x = 1/64$, $\Delta t = 0.0001$.

4.5.2 The Schemes DEI and DI

We now concentrate on the mixed scheme **DEI** (3.6.16) and the fully implicit scheme **DI** (3.6.12). We immediately remark that

- For $R \leq 0$ both schemes displayed the correct dynamics i.e. all solutions decayed to the trivial solution independent of the initial condition. Thus the numerics corroborate the results of Section 4.4.

- For $R > 0$ we tried the following forms of large initial data to test the existence of an absorbing ball independent of initial data $U^0 \in \mathbb{C}^J_{\text{per}}$.

  1. Constant initial data :

  $$U^0 = A(1, \cdots, 1) + iB(1, \cdots, 1)$$

  where $|A|, |B|$ was taken to be large ($|A|, |B| > 500$).

  2. Point–wise initial data :

  $$U^0 = (a, \cdots, a, A, a, \cdots, a) + (b, \cdots, b, B, b \cdots, b)$$

  where $|A|, |B|$ was taken to be large ($|A|, |B| > 1000$), with $|a|, |b| \ll |A|, |B|$.

3. Amplified sine waves :

$$U^0 = A\left(1, e^{2\pi i k\Delta x}, \cdots, e^{2\pi i k(J-1)\Delta x}\right),$$

where $k \in \mathbb{N}$ and $|A| > 1000$.

4. Oscillation on the grid :

$$U^0 = (A, -A, \cdots, A) + i(B, -B, \cdots, B),$$

with $(|A|, |B| > 1000)$.

For all of these types of initial data (for values of $|A|$, $|B|$ ranging up to 10000) the solutions of both schemes were seen to converge and become smooth. This confirms the results of the previous section and is a testament to the smoothing property. It should be noted that initial data that oscillates on the grid is not an approximation for an L^2 function. The numerical smoothing of this initial function indicates the Gevrey regularity we proved for the semi-discrete system in Section 4.2 and the continuous Gevrey result stated in Section 3.4.

For $R > 0$ we tested our numerical results against the continuous stability analysis of rotating waves outlined in Section 3.5.4, see Figures 3.1 and 3.2. The results we present are for $\nu = -\sqrt{3}$ and $\nu = \sqrt{3}$: other values of ν yielded analogous results.

First we consider $\nu = -\sqrt{3}$. From the continuous analysis for $R < 4\pi^2$ the spatially homogeneous rotating wave for the continuous system is stable and the trivial solution $U \equiv 0$ is unstable. In Figures 4.4–4.5 and 4.6–4.7 we have chosen initial conditions for the schemes **DEI** (3.6.16) and **DI** (3.6.12) close to the trivial solution (i.e. U^0 is taken to be the trivial solution plus a perturbation of order 10^{-6}). Both schemes exhibit the same behaviour : our solution leaves the trivial solution and is attracted to the stable spatially homogeneous wave. These figures should be compared to Figure 3.3 which shows the monochromatic wave (or Stokes solution) connecting the trivial solution and the spatially homogeneous wave for the continuous problem.

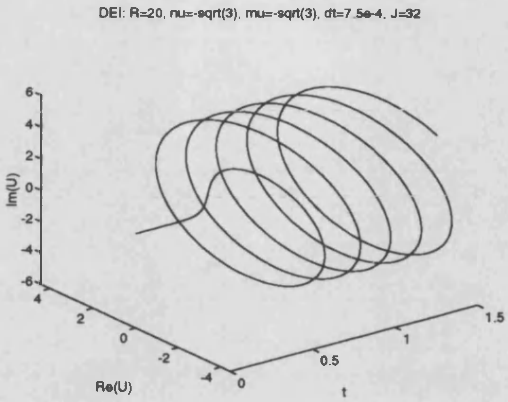

Figure 4.4: Connection from $U \equiv 0$ to stable spatially homogeneous wave for **DEI**: $R = 20$, $\nu = -\sqrt{3}$, $\Delta x = 1/32$, $\Delta t = 7.5 \times 10^{-4}$.

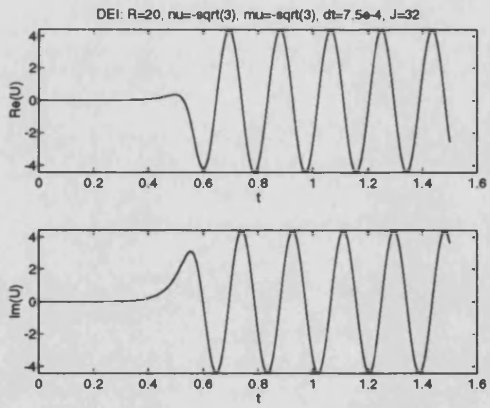

Figure 4.5: Connection from $U \equiv 0$ to stable spatially homogeneous wave, for **DEI**: $R = 20$, $\nu = -\sqrt{3}$, $\Delta x = 1/32$, $\Delta t = 7.5 \times 10^{-4}$.

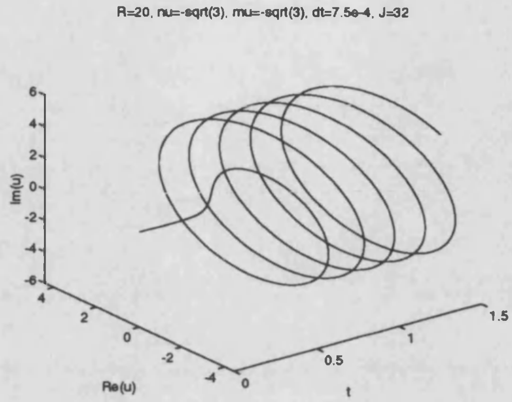

Figure 4.6: Connection from $U \equiv 0$ to stable spatially homogeneous wave for **DI**: $R = 20$, $\nu = -\sqrt{3}$, $\Delta x = 1/32$, $\Delta t = 7.5 \times 10^{-4}$.

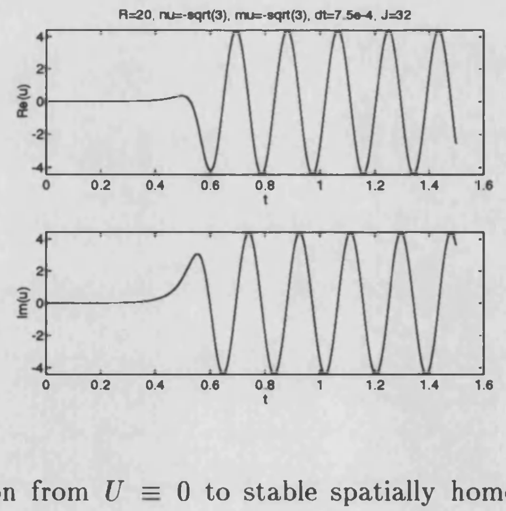

Figure 4.7: Connection from $U \equiv 0$ to stable spatially homogeneous wave for **DI**: $R = 20$, $\nu = -\sqrt{3}$, $\Delta x = 1/32$, $\Delta t = 7.5 \times 10^{-4}$.

For $R \geq 4\pi^2$ the dynamics becomes more complicated since as R increases the more solutions come into existence. For example at $R = 40$ there exists 2 rotating waves: the spatially homogeneous wave (which is linearly stable) and the first rotating wave which is unstable. In Figure 4.8 we have plotted a solution of **DEI** with large initial data: we see that it evolves to the spatially homogeneous rotating wave solution. Similar results are found for **DI**. In Figure 4.9 we have plotted for the scheme **DI** the evolution of an initial condition close to the 1st rotating wave. We see that it too is attracted to the spatially homogeneous solution, indicating the first rotating wave is unstable.

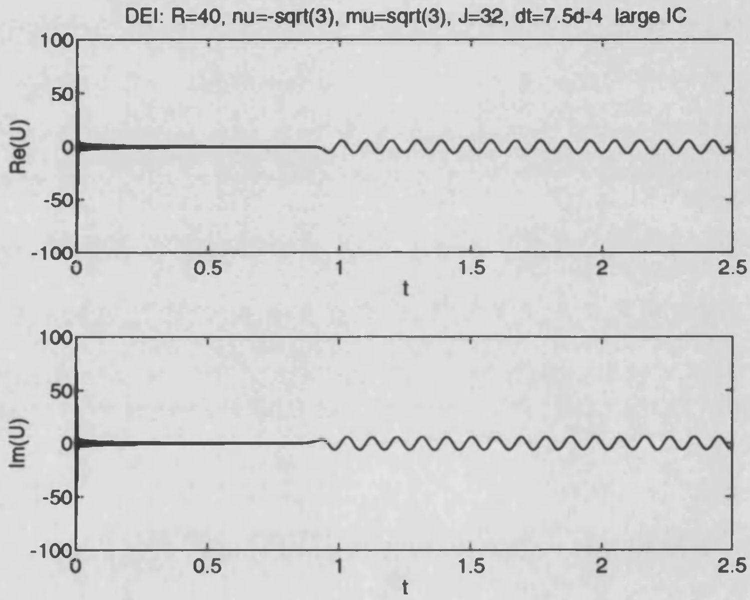

Figure 4.8: Large initial data for **DEI** attracted to spatially homogeneous wave. $R = 40$, $\nu = -\sqrt{3}$, $\Delta x = 1/32$, $\Delta t = 7.5 \times 10^{-4}$.

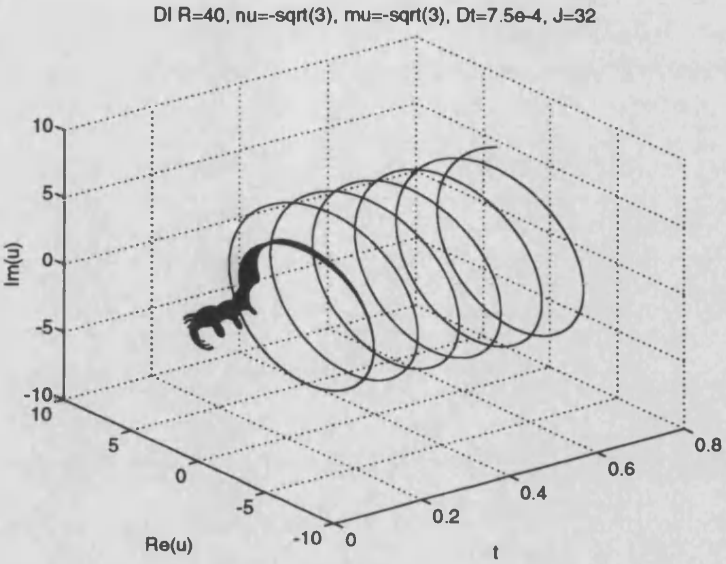

Figure 4.9: **DI** perturbed 1st rotating wave attracted to the spatially homogeneous wave. $R = 40$, $\Delta x = 1/32$, $\nu = -\sqrt{3}$, $\Delta t = 7.5 \times 10^{-4}$.

For a much larger value of R, $R = 1200$, and for the same large initial condition as in Figure 4.8 we find more complicated dynamical behaviour, this may be seen in Figures 4.10 and 4.11. Figure 4.10 shows the projection of $U(t)$ through time onto the complex plane. The initial condition is the large outer circle, then the dynamics evolves onto the smaller circle with approximate amplitude 22. Figure 4.11 shows the evolution in time of the point $10\Delta x \approx 0.3$ on the interval $[0, 1]$. In Figure 4.12 we have calculated the Lyapunov exponent associated with the solution shown in Figures 4.10 and 4.11 using the numerical scheme we discussed in section 2.1.1. We note that the exponent is negative, and so although the dynamics is more complicated we do not have any exponential stretching which we associate with chaos. Indeed this solution appears to correspond to a solution of higher wave number (approximately 13). We note that similar results are found for the scheme **DI** (3.6.12).

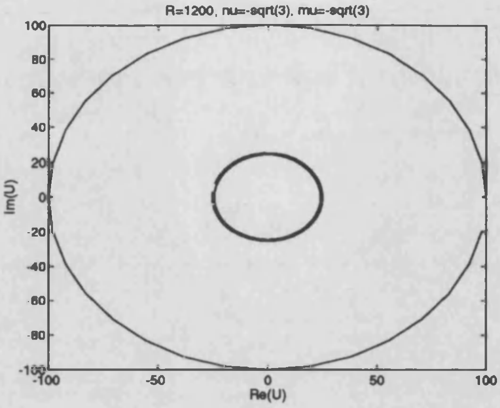

Figure 4.10: Projection onto complex plane for $R = 1200, \nu = -\sqrt{3}$, $\Delta x = 1/32$, $\Delta t = 7.5 \times 10^{-4}$.

We now consider $\nu = \sqrt{3}$.

For $R < 4\pi^2$ then we are in the same position as for $\nu = -\sqrt{3}$, the trivial solution for the continuous system is unstable whereas the spatially homogeneous solution is stable. This is exactly what we find numerically for the schemes **DEI** and **DI**. For example in Figure 4.13 we have found the connection from the trivial solution to spatially homogeneous rotating wave solution for **DI**.

For $R \geq 4\pi^2$ we see from Figure 3.2 that the stability structure of the rotating waves is quite different. The results we present below are for the scheme **DI** however completely analogous results are found for the scheme **DEI**.

With the same initial condition as in Figures 4.8,4.10 and 4.11 we have computed the solution from **DI** for $R = 120$ and $R = 1200$. This is plotted in Figures 4.14 and 4.15 for $R = 120$ and for $R = 1200$ in Figures 4.16 and 4.17. In each case we have calculated the associated largest Lyapunov using the method discussed in Section 2.1.1. In Figure 4.18 we have plotted the largest exponent for the case $R = 120$ and in Figure 4.19 the largest exponent for the case $R = 1200$. For both $R = 120$ and $R = 1200$ the Lyapunov exponent appears to be positive, and since the solutions appear to be stable these are possibly "chaotic" solutions.

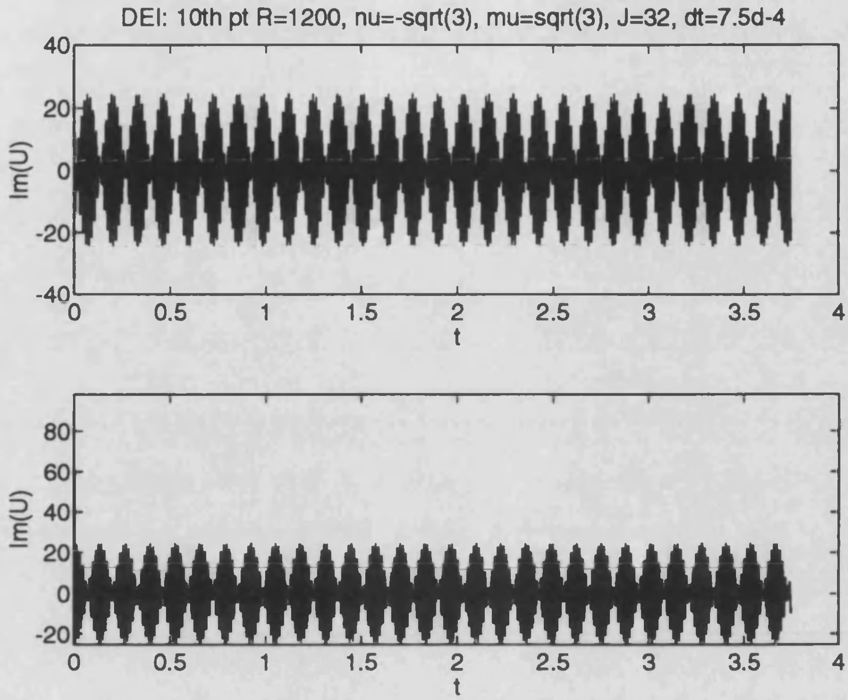

Figure 4.11: Real (top) and imaginary (bottom) parts of solution through time: $R = 1200$, $\nu = -\sqrt{3}$, $\Delta x = 1/32$, $\Delta t = 7.5 \times 10^{-4}$.

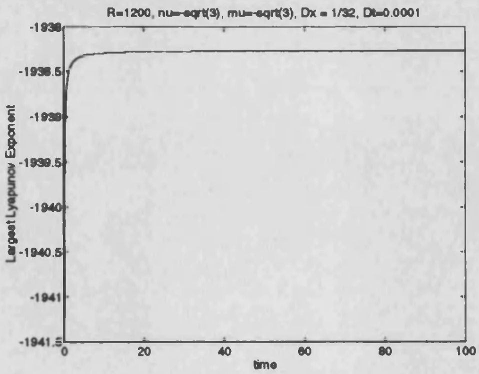

Figure 4.12: First Lyapunov exponent: $R = 1200$, $\nu = -\sqrt{3}$, $\Delta x = 1/32$, $\Delta t = 10^{-4}$.

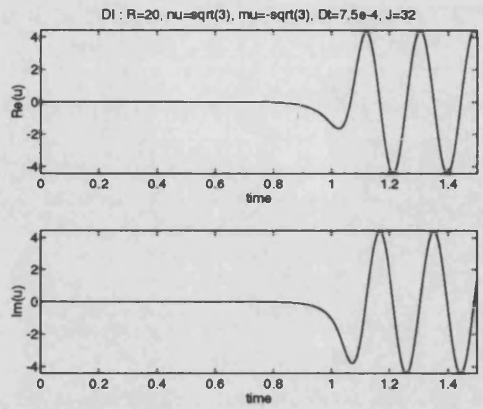

Figure 4.13: Connection from $U \equiv 0$ to the first rotating wave solution for **DI**: $R = 20$, $\nu = \sqrt{3}$, $\Delta x = 1/32$, $\Delta t = 7.5 \times 10^{-4}$.

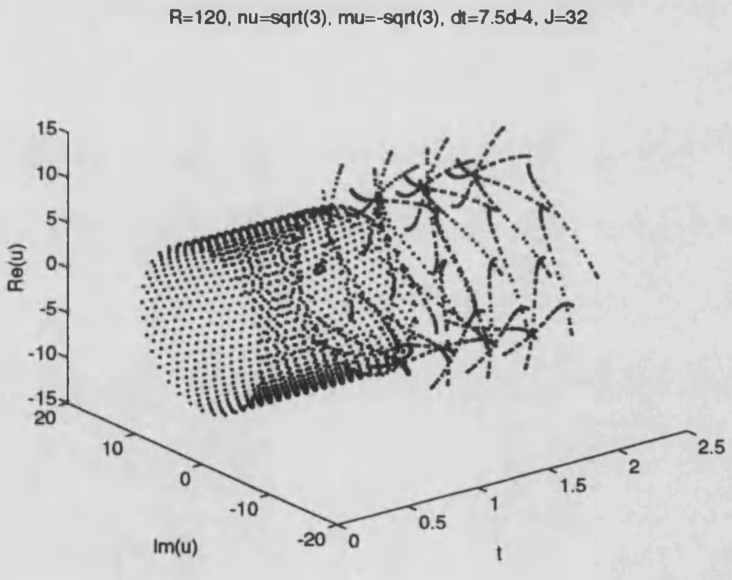

Figure 4.14: Solution for $R = 120$, $\nu = \sqrt{3}$, $\Delta x = 1/32$, $\Delta t = 7.5 \times 10^{-4}$.

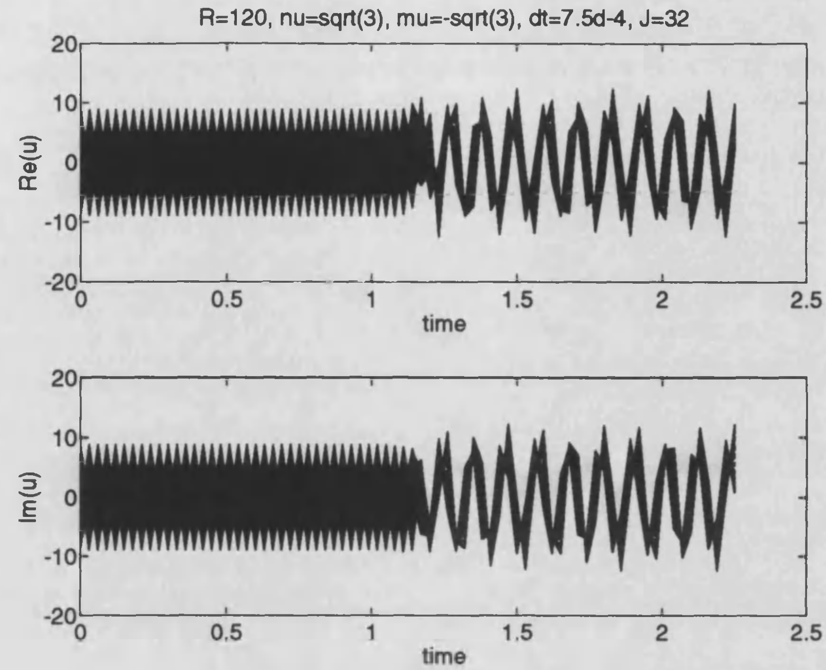

Figure 4.15: Solution for $R = 120$, $\nu = \sqrt{3}$, $\Delta x = 1/32$, $\Delta t = 7.5 \times 10^{-4}$.

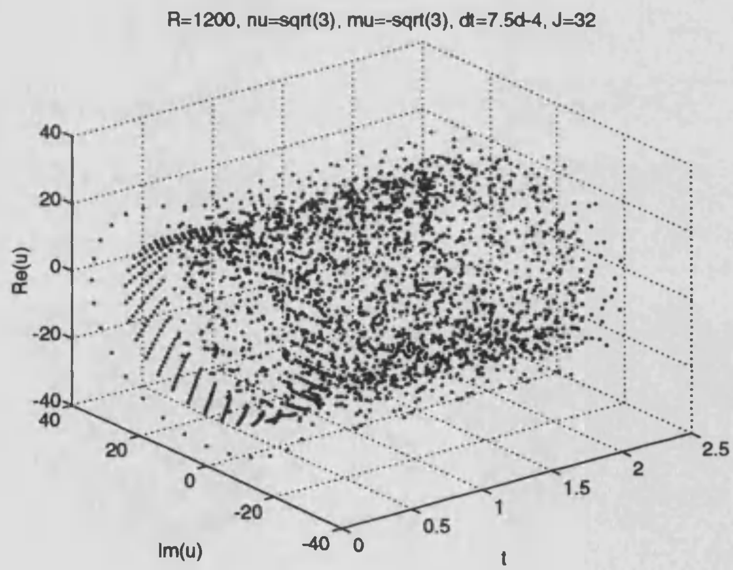

Figure 4.16: Solution for $R = 1200$, $\nu = \sqrt{3}$, $\Delta x = 1/32$, $\Delta t = 7.5 \times 10^{-4}$.

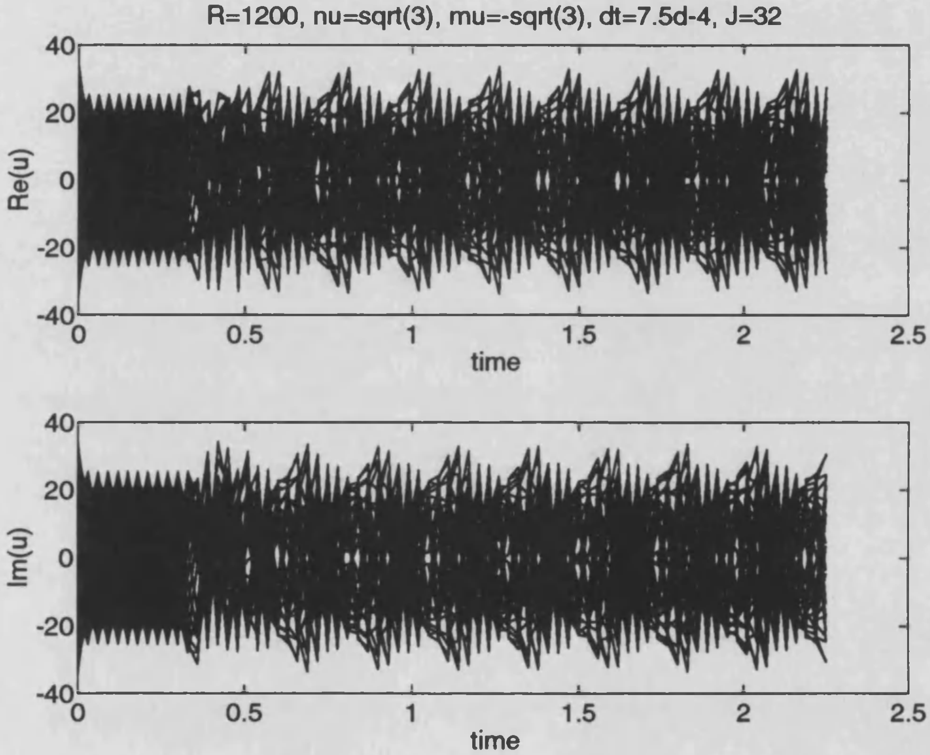

Figure 4.17: $R = 1200$, $\nu = \sqrt{3}$, $\Delta x = 1/32$, $\Delta t = 7.5 \times 10^{-4}$.

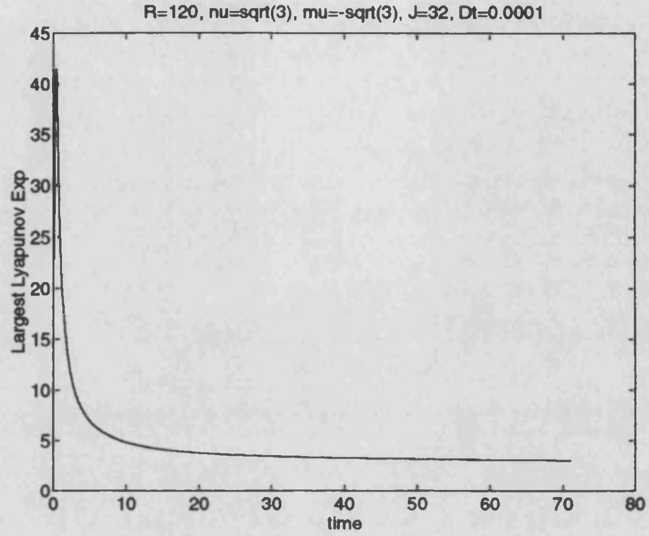

Figure 4.18: First Lyapunov exponent for $R = 120$, $\nu = \sqrt{3}$, $\Delta x = 1/32$, $\Delta t = 10^{-4}$.

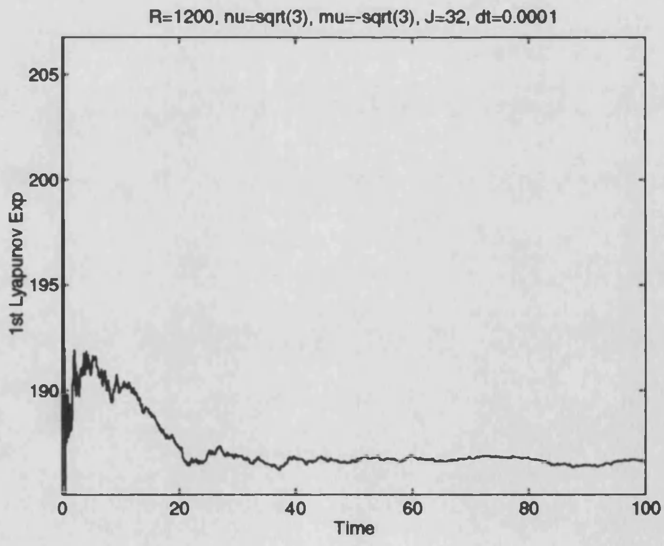

Figure 4.19: First Lyapunov exponent for $R = 1200$, $\nu = \sqrt{3}$, $\Delta x = 1/32$, $\Delta t = 10^{-4}$.

4.5.3 Dependence of Δt on Δx for DEI

In Theorem 4.4.8 we prove the existence of an absorbing ball $\mathcal{B}_0(\rho_0) \subset L^2_{\Delta x}$, with ρ_0 independent of U^0, for the finite difference scheme **DEI** (3.6.16) with a restriction (4.4.21) on the time step Δt in terms of the spatial step Δx. In this section we present numerical evidence that this restriction is a product of the analysis and not a true reflection on the scheme **DEI**.

To achieve this we take initial data $|U^0|_{L^2_{\Delta x}}$ with $L^2_{\Delta x}$ norm of order 100 and fix the parameters R, ν, μ and a total integration time $T > 0$. Then for $\Delta x = 1/J, (J \in \mathbb{N})$ we vary Δt integrating to time T in each case.

In Figure 4.20 we see the results of this process for $R = 20$, $\nu = \mu = -\sqrt{3}$, initial data U^0 with norm $|U^0|_{L^2_{\Delta x}} > 100$ and integrating to a fixed time $T = 4$ at which the solution has converged to the spatially homogeneous rotating wave. This is a 3D plot showing the norm of the computed solution squared versus $\log(\Delta t)$ and J where the values of J taken were $J = 8, 16, 32, 64$ and 128. The computed norms shown for $J = 8, 16, 32, 64$ and 128 differ at each Δt only in the eighth decimal place. In Figure 4.21 we have repeated the process but for $R = 40$, $\nu = \sqrt{3}$ and $\mu = -\sqrt{3}$ with the same initial data as in Figure 4.8 but integrating until $T = 10$. The values of J taken were again $J = 8, 16, 32, 64$ and 128. The computed norms shown for $J = 8, 16, 32, 64$ and 128 differ at each Δt only in the eighth decimal place. Both figures 4.20 and 4.21 indicate that Δt is *independent* of Δx.

By a simple examination of the continuous rotating wave solutions (3.5.76) we see that the continuous L^2 norm of the spatially homogeneous rotating wave U_0 solution is given by

$$|U_0|^2_{L^2} = R.$$

In Figures 4.22 and 4.23 we have plotted the log of the error,

$$\log(|U_0|^2_{L^2} - |U_0^c|^2_{L^2_{\Delta x}}),$$

versus $\log(\Delta t)$ where U_0^c is the computed approximation at time T. In fact we have plotted in Figures 4.22 and 4.23 results for $J = 8, 16, 32, 64, 128$, but the difference between them is not discernable since the error was of order 10^{-5} and smaller. This shows that we find a good approximation with quite large values of Δx and that Δt

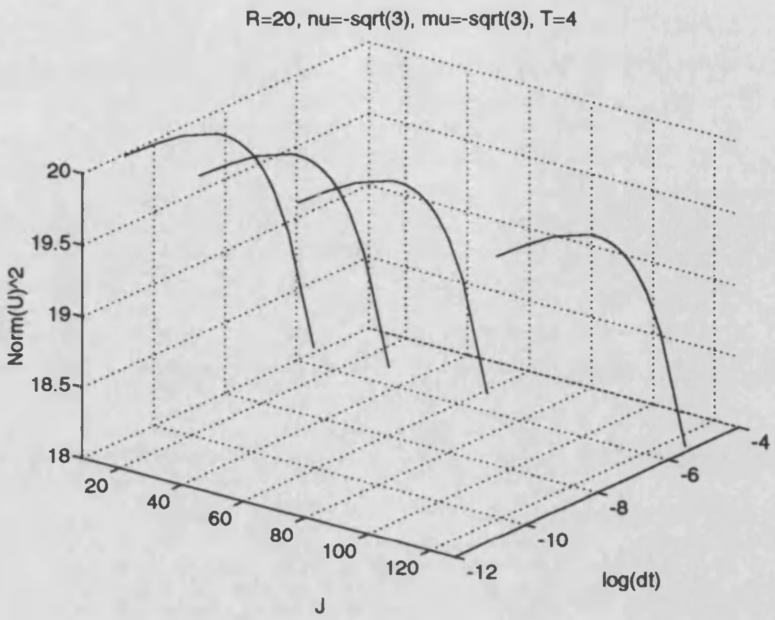

Figure 4.20: Independence of Δt on Δx for $R = 20$, $\nu = \mu = \sqrt{3}$, $J = 8, 16, 32, 64, 128$.

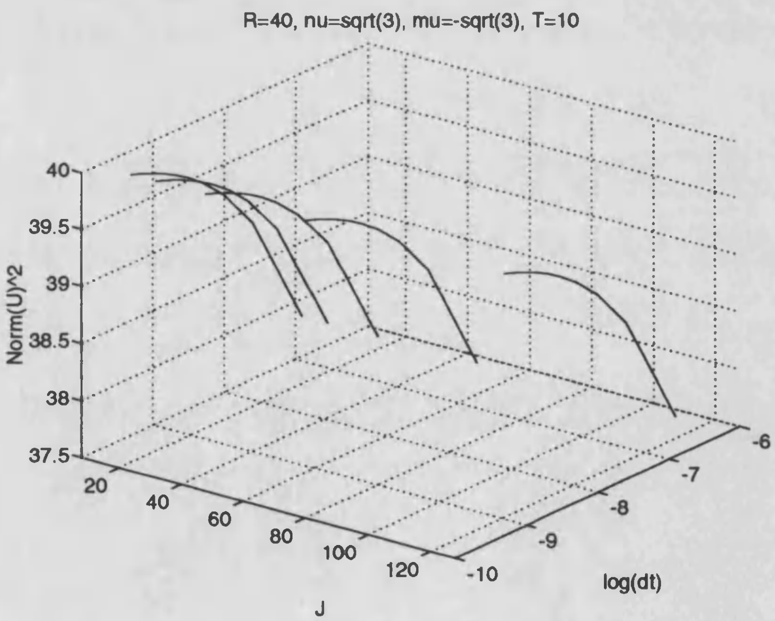

Figure 4.21: Independence of Δt on Δx for $R = 40$, $\nu = \sqrt{3}$, $\mu = -\sqrt{3}$, $J = 8, 16, 32, 64, 128$.

does not appear to be restricted in anyway by Δx. From the figures we see that the error decreases linearly in Δt, this is as we expect since the scheme is of order Δt. The values of R, ν, μ and T are as in Figures 4.20 and 4.21 for Figures 4.22 and 4.23 respectively. Comparing figures 4.22 and 4.23 we see that the error and the value of Δt required for a given accuracy depends on the parameters.

The importance of picking Δt sufficiently small depending on the parameters is further illustrated in Figure 4.24 where for large initial data we have plotted the norm of computed solution at $T = 25$ for various values of Δt. The value of Δx was fixed at $\Delta x = 1/32$, and the other parameters were $R = 50$, $\nu = \sqrt{3}$ and $\mu = -\sqrt{3}$. In Figure 4.25 we have plotted the norm of solution U at $T = 25$ for 4 values of Δt and $J = 8$, 16, 32, 64, 128, this illustrates that in this case too Δt is independent of Δx.

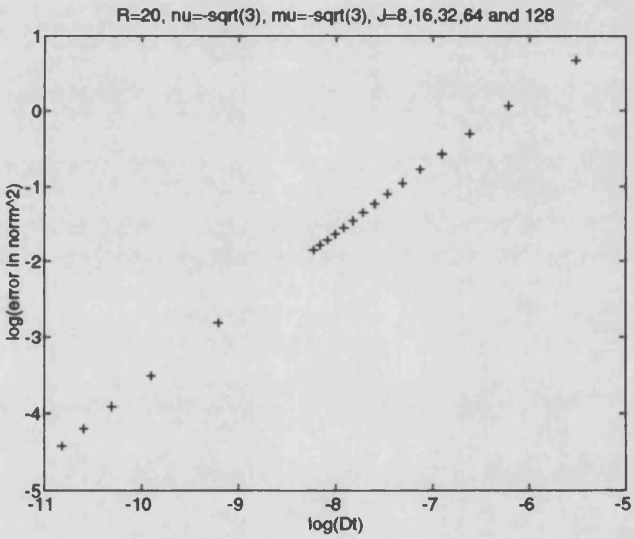

Figure 4.22: Error in norm for $R = 20$, $\nu = \mu = -\sqrt{3}$, $J = 8, 16, 32, 64, 128$.

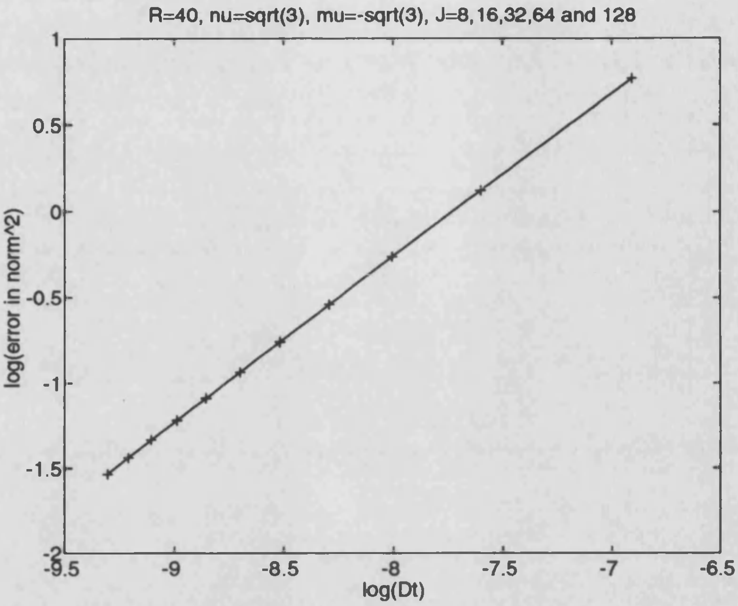

Figure 4.23: Error in norm for $R = 40$, $\nu = \sqrt{3}$, $\mu = -\sqrt{3}$, $J = 8, 16, 32, 64, 128$.

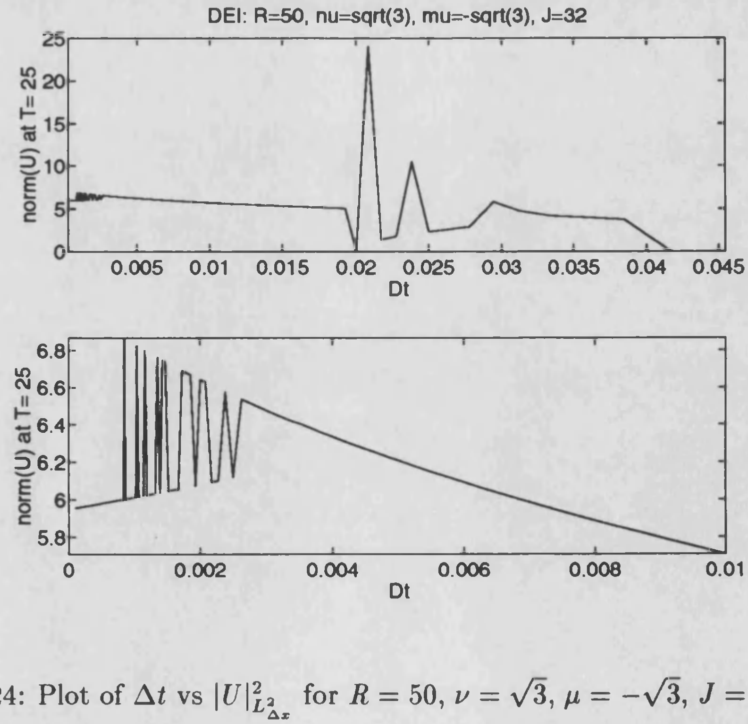

Figure 4.24: Plot of Δt vs $|U|^2_{L^2_{\Delta x}}$ for $R = 50$, $\nu = \sqrt{3}$, $\mu = -\sqrt{3}$, $J = 32$, $T = 25$.

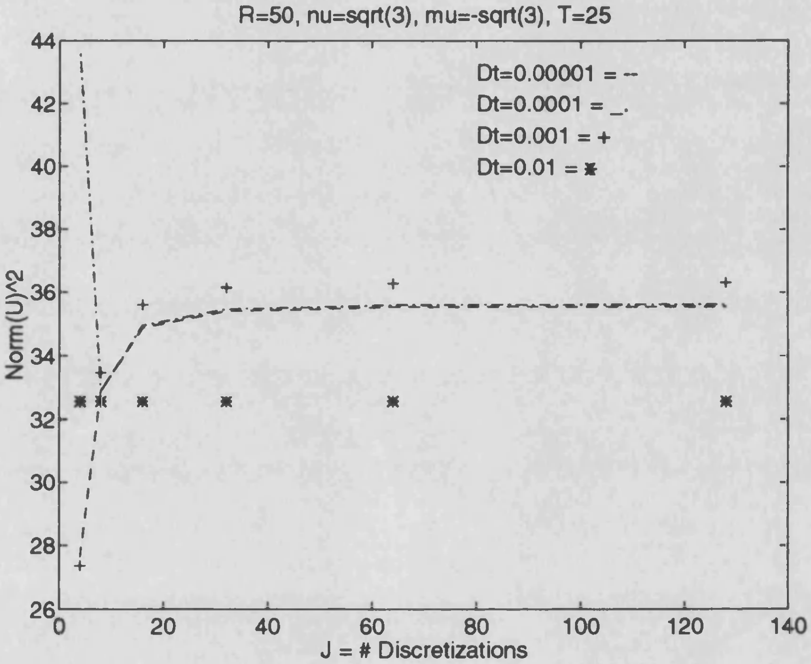

Figure 4.25: Plot of J vs $|U|^2_{L^2_{\Delta x}}$ for $\Delta t = 10^{-2}$, $\Delta t = 10^{-4}$, $\Delta t = 10^{-5}$ and $\Delta t = 10^{-6}$ ($R = 50$, $\nu = \sqrt{3}$, $\mu = -\sqrt{3}$, $J = 8, 16, 32, 64, 128$).

Chapter 5

Low Dimensional Dynamics

5.1 An Inertial Manifold for the Semi–Discrete System

In Section 3.5.1 we showed the existence of an inertial manifold for the continuous Ginzburg–Landau equation (3.1.1) and hence that the interesting dynamics for the problem took place on a finite dimensional manifold. Clearly it is desirable that numerical schemes for (3.1.1) should have an inertial manifold and that manifold should approximate the inertial manifold for the full problem.

We shall prove that, provided the spatial resolution is high enough, the semi–discrete Ginzburg–Landau equation (3.6.10) admits an inertial manifold - that is a manifold which is exponentially attracting and positively invariant of dimension less than J, the dimension of the semi–discrete system. Note that it is necessarily finite dimensional since (3.6.10) is a finite dimensional system. The requirement that the number of grid points J be sufficiently large (or equivalently the mesh size be sufficiently small) arises naturally as we must assure that the dimension of the discrete system is at least as large as that of the inertial manifold of the continuous problem and we are trying to approximate an inertial manifold of a certain dimension.

We commence by defining the projection $\mathcal{P}_m$ onto low Fourier modes

$$\mathcal{P}_m : L^2_{\Delta x} \to \mathrm{Sp}\,\{\psi_{-m}, \cdots, \psi_0, \cdots, \psi_m\}\,;$$

the projection $\mathcal{Q}_m$ onto high modes

$$\mathcal{Q}_m = I - \mathcal{P}_m$$

along with the spaces

$$Y_{\Delta x} := \mathcal{P}_m L^2_{\Delta x} \quad \text{and} \quad Z_{\Delta x} := \mathcal{Q}_m L^2_{\Delta x}.$$

In an analogous manner to the continuous case we have

$$L^2_{\Delta x} = Y_{\Delta x} \oplus Z_{\Delta x}.$$

As an illustration consider $u \in L^2_{\Delta x}$ given by the Fourier series expansion

$$u = \sum_{k=-J/2\rceil}^{J/2\rfloor} a_k \psi_k,$$

then,

$$\mathcal{P}_m u = \sum_{k=-m}^{m} a_k \psi_k, \quad \text{and} \quad \mathcal{Q}_m u = \sum_{k=m+1}^{J/2\rceil} a_k \psi_k + \sum_{k=-(J/2\rfloor)}^{-(m+1)} a_k \psi_k.$$

We shall again invoke Theorem 3.5.1 to prove the existence of an inertial manifold, to which end we prove a few preliminary lemmas. The following lemma provides upper and lower bounds on the eigenvalues $\{\lambda_k\}$ given in Lemma 3.7.1 of the discrete linear operator $M^{-1}A$.

Lemma 5.1.1 *The eigenvalues $\{\lambda_k\}$ of $M^{-1}A$ satisfy:*

$$\lambda_k^{1/2} \leq 2k\pi, \quad \textit{and } \lambda_k^{1/2} \geq 2\left(k\pi - \frac{k^3\pi^3\Delta x^2}{6}\right).$$

Proof Both inequalities are immediate after an application of Taylor's theorem to $\sin(k\pi\Delta x)$ with a Lagrange remainder term. □

Lemma 5.1.2 *The eigenvalues $\{\lambda_k\}$, of $M^{-1}A$ satisfy*

$$\lambda_{k+1}^{1/2} \leq \lambda_k^{1/2} + \lambda_1^{1/2}$$

and

$$\lambda_{k+1}^{1/2} \geq \lambda_k^{1/2}(1 - \pi^2\Delta x^2/2) + \lambda_1^{1/2}(1 - k^2\pi^2\Delta x^2/2).$$

Hence we find the inequality

$$\lambda_{k+1} \geq \lambda_k + \lambda_1 + 2\lambda_k^{1/2}\lambda_1^{1/2} - \pi^2\Delta x^2\left(\lambda_k + \lambda_1 k^2 + (1+k^2)\lambda_k^{1/2}\lambda_1^{1/2}\right).$$

Proof Recall that the eigenvalues $\{\lambda_k\}$ of $M^{-1}A$ are given by

$$\lambda_k = \frac{4}{\Delta x^2}\sin^2(k\pi\Delta x).$$

Thus

$$\lambda_{k+1}^{1/2} = \frac{2}{\Delta x}\sin(k\pi\Delta x + \pi\Delta x)$$

and using a trigonometric identity we have

$$\lambda_{k+1}^{1/2} = \lambda_k^{1/2}\cos(\pi\Delta x) + \lambda_1^{1/2}\cos(k\pi\Delta x).$$

Taylor's theorem with Lagrange form of remainder gives

$$\cos(k\pi\Delta x) = 1 - \frac{\pi^2\Delta x^2}{2}\cos(t), \quad \text{for some } t \in (0, k\pi\Delta x).$$

Since

$$\cos(k\pi\Delta x) \leq 1 \text{ and } \cos(k\pi\Delta x) \geq 1 - \frac{k^2\pi^2\Delta x^2}{2}$$

we find

$$\lambda_{k+1}^{1/2} \leq \lambda_k^{1/2} + \lambda_1^{1/2}$$

and

$$\lambda_{k+1}^{1/2} \geq \lambda_k^{1/2}(1 - \pi^2\Delta x^2/2) + \lambda_1^{1/2}(1 - k^2\pi^2\Delta x^2/2). \tag{5.1.1}$$

Thus we have the first two inequalities.

We now proceed to establish the third. From (5.1.1) :

$$\begin{aligned}\lambda_{k+1} \;\; \geq \;\; & \lambda_k\left(1 - \frac{\pi^2\Delta x^2}{2}\right)^2 + \lambda_1\left(1 - \frac{k^2\pi^2\Delta x^2}{2}\right)^2 \\ & + 2\lambda_k^{1/2}\lambda_1^{1/2}\left(1 - \frac{\pi^2\Delta x^2}{2}\right)\left(1 - \frac{k^2\pi^2\Delta x^2}{2}\right).\end{aligned}$$

Multiplying out the brackets and neglecting positive terms of order $O(\Delta x^4)$ on the right–hand side we obtain the final inequality.□

We now prove the spectral gap condition for the discrete linear operator $\widetilde{A}$.

Lemma 5.1.3 (Spectral Gap) *Given K_1, $K_2 > 0$ there exists $m \in \mathbb{Z}$ and $\Delta x_0 > 0$, such that for all $\Delta x < \Delta x_0$, the eigenvalues of $\widetilde{A}$ satisfy*

$$\widetilde{\lambda}_{m+1} \geq K_1 \;\; \textit{and} \;\; \widetilde{\lambda}_{m+1} - \widetilde{\lambda}_m \geq K_2.$$

Proof First we fix m and Δx_0 and note that result is straightforward for $K_1 \leq 1$, so we consider $K_1 > 1$. Let

$$m > \max\left\{\frac{\sqrt{K_1 - 1}}{2\pi} - 1, \frac{K_2}{8\pi^2} - \frac{1}{2}\right\} \tag{5.1.2}$$

and

$$\Delta x_0 = \min\left\{\frac{1 - K_1 + 4\pi^2(m+1)^2}{4\pi^4}, \frac{8\pi^2 m + 4\pi^2 - K_2}{4\pi^4}, \frac{3}{(m+1)^4}\right\}. \tag{5.1.3}$$

By the choice of m we have that

$$1 - K_1 + 4\pi^2(m+1)^2 > 0 \tag{5.1.4}$$

and

$$8\pi^2 m + 4\pi^2 - K_2 > 0 \tag{5.1.5}$$

whence Δx_0 is well defined.

By the choice of Δx_0 we have that

$$m > \frac{\sqrt{K_1 + 4\pi^4\Delta x_0 - 1}}{2\pi} - 1 \tag{5.1.6}$$

and

$$m > \frac{1}{8\pi^2}\left(K_2 + 4\pi^4\Delta x_0 - 4\pi^2\right), \tag{5.1.7}$$

which by Lemma 3.5.1 implies that

$$\widetilde{\Lambda}_{m+1} > K_1 + 4\pi^4\Delta x_0 \tag{5.1.8}$$

and

$$\widetilde{\Lambda}_{m+1} - \widetilde{\Lambda}_m > K_2 + 4\pi^4\Delta x_0. \tag{5.1.9}$$

We further note that $\Delta x_0 \leq 3/m^4 < 1/m \Longrightarrow m < J$ (for $m \geq 1$) and so the dimension J of the semi–discrete problem is high enough that λ_m and λ_{m+1} are well defined.

Clearly we have

$$\widetilde{\lambda}_{m+1} = \widetilde{\Lambda}_{m+1} - (\widetilde{\Lambda}_{m+1} - \widetilde{\lambda}_{m+1}),$$

and by equation (5.1.8) and Lemma 5.1.1 we see that

$$\begin{aligned}
\widetilde{\lambda}_{m+1} &\geq K_1 + 4\pi^4\Delta x_0 - (\widetilde{\Lambda}_{m+1} - \widetilde{\lambda}_{m+1}) \\
&\geq K_1 + 4\pi^4\Delta x_0 - \frac{4\pi^4(m+1)^4\Delta x^2}{3} + \frac{(m+1)^6\pi^6\Delta x^4}{9} \\
&\geq K_1 + 4\pi^4\Delta x_0 - 4\pi^4\Delta x_0 \\
&= K_1.
\end{aligned}$$

Furthermore

$$\widetilde{\lambda}_{m+1} - \widetilde{\lambda}_m = \widetilde{\Lambda}_{m+1} - \widetilde{\Lambda}_m + (\widetilde{\lambda}_{m+1} - \widetilde{\Lambda}_{m+1}) - (\widetilde{\lambda}_m - \widetilde{\Lambda}_m),$$

so by Lemma 5.1.1, equation (5.1.9) and the fact that $\widetilde{\lambda}_m - \widetilde{\Lambda}_m \leq 0$ for all $m \in \mathbb{Z}$,

$$\begin{aligned} \widetilde{\lambda}_{m+1} - \widetilde{\lambda}_m &= K_2 + 4\pi^4 \Delta x_0 - \frac{4\pi^4 m^4 \Delta x^2}{3} + \frac{m^6 \pi^6 \Delta x^4}{9} \\ &\geq K_2 + 4\pi^4 \Delta x_0 - 4\pi^4 \Delta x_0 \\ &= K_2. \quad \square \end{aligned}$$

Note: By the choice of Δx_0 we have ensured that the break in the spectrum is the same for the finite difference approximation and the partial differential equation, this can be seen by comparing Lemmas 5.1.3 and 3.5.1.

Lemma 5.1.4 *Given K_1, $K_2 > 0$ there exists $m \in \mathbb{Z}$ and $\Delta x_0 > 0$ such that for all $\Delta x < \Delta x_0$ the eigenvalues λ_k of $M^{-1}A$ satisfy the spectral gap property, i.e.*

$$\lambda_{m+1} \geq K_1 \quad \textit{and} \ \lambda_{m+1} - \lambda_m \geq K_2.$$

Proof We fix m and Δx_0 and note the result is straightforward for $K_1 \leq 1$, and so consider we consider $K_1 > 1$. Let

$$m > \max\left\{\frac{\sqrt{K_1 - 1}}{2\pi}, \frac{K_2}{8\pi^2} - \frac{1}{2}\right\} \tag{5.1.10}$$

and

$$\Delta x_0 = \min\left\{\frac{-K_1 + 4\pi^2(m+1)^2}{4\pi^4}, \frac{8\pi^2 m + 4\pi^2 - K_2}{4\pi^4}, \frac{3}{m^4}\right\}. \tag{5.1.11}$$

The remainder of the lemma is proved in exactly the same manner as Lemma 5.1.3 with $\widetilde{\Lambda}_{m+1}$ replaced by Λ_{m+1} and $\widetilde{\Lambda}_m$ replaced by Λ_m. The details are omitted. $\square$

We now turn our attention to proving the relevant Lipschitz conditions for the nonlinearity.

Lemma 5.1.5 *Let $\rho > 0$ be given. Then there exists $C = C(\rho) > 0$ such that the non-linear function F given by (3.6.11) satisfies*

$$\|F(u)\|_{H^1_{\Delta x}} \leq C \quad \forall u \in \mathcal{B}_1(\rho)$$

and

$$\|F(u) - F(v)\|_{H^1_{\Delta x}} \leq C\|u - v\|_{H^1_{\Delta x}} \quad \forall u, v \in \mathcal{B}_1(\rho).$$

Proof Since by the definition of F we have that $F(0) = 0$, it is sufficient to prove the second inequality alone.

Note that

$$\|F(u) - F(v)\|_{H^1_{\Delta x}}^2 = |F(u) - F(v)|_{L^2_{\Delta x}}^2 + \|F(u) - F(v)\|_1^2$$

and by Lemma 3.7.21 and the simple inequality (3.5.47) ($(a+b)^2 \leq 2a^2 + 2b^2\ \forall a, b \in \mathbb{R}$);

$$\begin{aligned}
|F(u) - F(v)|_{L^2_{\Delta x}}^2 &= \sum_{j=0}^{J-1} \Delta x \left|\tilde{R}(U_j - V_j) - (1 + i\mu)\left(|U_j|^2 U_j - |V_j|^2 V_j\right)\right|^2 \\
&\leq 2|\tilde{R}|^2 |U - V|_{L^2_{\Delta x}}^2 + 2(1 + \mu^2)^2 C |U - V|_{L^2_{\Delta x}}^2 \\
&= C|U - V|_{L^2_{\Delta x}}^2 .
\end{aligned}$$

All that remains is to remark that by Lemma 3.7.22

$$\begin{aligned}
\|F(U) - F(V)\|_1^2 &= \sum_{j=0}^{J-1} \Delta x \left|\tilde{R}\delta_+(U_j - V_j) - (1 + i\mu)\delta_+(|U_j|^2 U_j - |V_j|^2 V_j)\right|^2 \\
&\leq 2|\tilde{R}|^2 \|U - V\|_1^2 + 2(1 + \mu^2)\|G(|U|^2)U - G(|V|^2)V\|_1^2 \\
&\leq C\|U - V\|_1^2,
\end{aligned}$$

and the lemma is proved.□

We now invoke Theorem 3.5.1 to prove the existence of an inertial manifold.

Theorem 5.1.1 *There exists an inertial manifold $\mathcal{M}_{\Delta x}$ for the semi-discrete Ginzburg–Landau equation* (3.1.1) *which may be represented as a graph of $\Phi : Y_{\Delta x} \to Z_{\Delta x}$ within the absorbing ball $\mathcal{B}_1(\rho_1)$.*

Proof By Lemma 5.1.3, Lemma 5.1.5, Theorem 4.1.3 and Remark 3.7.1 all the assumptions for Theorem 3.5.1 are satisfied. □

5.1.1 Cone Condition for the Semi–Discrete System.

In Theorem 3.5.3 we stated the result due to [41] that the cone condition holds for the Ginzburg–Landau equation for two solutions on the global attractor. In this section we prove the semi–discrete equivalent to that theorem. First, however, a preliminary lemma.

Lemma 5.1.6 *Given any $U \in L^2_{\Delta x}$,*

$$\|\mathcal{P}_m U\|_1^2 \le \lambda_m |\mathcal{P}_m U|^2_{L^2_{\Delta x}}$$

and

$$\|\mathcal{Q}_m U\|_1^2 \ge \lambda_{m+1} |\mathcal{Q}_m U|^2_{L^2_{\Delta x}}.$$

Proof Let U have Fourier coefficients $\{a_k\}$ and for convenience write $p = \mathcal{P}_m U$ and $q = \mathcal{Q}_m U$.

Then

$$\|p\|_1^2 = \sum_{k=-m}^{m} \lambda_k |a_k|^2 \le \lambda_m \sum_{k=-m}^{m} |a_k|^2 = \lambda_m |p|^2_{L^2_{\Delta x}}$$

and

$$\begin{aligned}
\|q\|_1^2 &= \sum_{k=-\lceil J/2 \rceil}^{-m} \lambda_k |a_k|^2 + \sum_{k=m+1}^{\lfloor J/2 \rfloor} \lambda_k |a_k|^2 \\
&\ge \lambda_{m+1} \sum_{k=-\lceil J/2 \rceil}^{-m} |a_k|^2 + \lambda_{m+1} \sum_{k=m+1}^{\lfloor J/2 \rfloor} |a_k|^2 \\
&= \lambda_{m+1} |q|^2_{L^2_{\Delta x}}. \qquad \square
\end{aligned}$$

Theorem 5.1.2 *Let $U^0, V^0 \in \mathcal{B}_1(\rho_1)$ where ρ_1 is given in Theorem 4.1.3 and define B by*

$$B := \max_{t>0} \left\{ |U(t)|^2_{L^\infty}, |V(t)|^2_{L^\infty} \right\}.$$

Let m and Δx_0 be found from Lemma 5.1.4 so that

$$\lambda_m > \max\left\{ R, \frac{9B^2(1+\mu^2)}{4\lambda_1} \right\} \tag{5.1.12}$$

and let Δx_1 be given by

$$\Delta x_1^2 \le \left\{ \sqrt[3]{\frac{3}{\pi^2}}, \frac{1}{2\pi\sqrt{2(2m^2 + (1+m^2)m)}}, \Delta x_0 \right\}. \tag{5.1.13}$$

Then for all $\Delta x < \Delta x_1$ solutions $U(t), V(t)$ satisfy the cone condition for $t > 0$ with cone given by

$$\mathcal{C}_{m,1} = \left\{ W \in L^2_{\Delta x} : |\mathcal{Q}_m W|^2_{L^2_{\Delta x}} \le |\mathcal{P}_m W|^2_{L^2_{\Delta x}} \right\}.$$

Proof

Let $U(t), V(t)$ be two solutions to the semi-discrete Ginzburg–Landau equation (3.1.1). Then the difference $U(t) - V(t)$ satisfies

$$\frac{d}{dt}(U-V) = R(U-V) - (1+i\nu)M^{-1}A(U-V) - (1+i\mu)\left\{G(|U|^2)U - G(|V|^2)V\right\}.$$

The projection $p(t) := \mathcal{P}_m(U(t)-V(t))$ onto low wave numbers satisfies :

$$\frac{dp}{dt} = Rp - (1+i\nu)M^{-1}Ap - (1+i\mu)\mathcal{P}_m\left\{G(|U|^2)U - G(|V|^2)V\right\} \tag{5.1.14}$$

and the projection $q(t) := \mathcal{Q}_m(U(t)-V(t))$ onto high wave numbers satisfies :

$$\frac{dq}{dt} = Rq - (1+i\nu)M^{-1}Aq - (1+i\mu)\mathcal{Q}_m\left\{G(|U|^2)U - G(|V|^2)V\right\}. \tag{5.1.15}$$

If we take the inner-product of (5.1.14) with p and (5.1.15) with q and take the real part we find

$$\frac{1}{2}\frac{d|p|^2_{L^2_{\Delta x}}}{dt} \geq R|p|^2_{L^2_{\Delta x}} - \|p\|_1^2 - \operatorname{Re}\left\{(1+i\mu)\sum_{j=0}^{J-1}\Delta x\overline{p_j}\mathcal{P}_m(|U_j|^2U_j - |V_j|^2V_j)\right\}$$

and,

$$\frac{1}{2}\frac{d|q|^2_{L^2_{\Delta x}}}{dt} \leq R|q|^2_{L^2_{\Delta x}} - \|q\|_1^2 - \operatorname{Re}\left\{(1+i\mu)\sum_{j=0}^{J-1}\Delta x\overline{q_j}\mathcal{Q}_m(|U_j|^2U_j - |V_j|^2V_j)\right\}.$$

Apply Lemma 5.1.6

$$\frac{1}{2}\frac{d|p|^2_{L^2_{\Delta x}}}{dt} \geq R|p|^2_{L^2_{\Delta x}} - \lambda_m|p|^2_{L^2_{\Delta x}} - \operatorname{Re}\left\{(1+i\mu)\sum_{j=0}^{J-1}\Delta x\overline{p_j}\mathcal{P}_m(|U_j|^2U_j - |V_j|^2V_j)\right\}$$

$$\frac{1}{2}\frac{d|q|^2_{L^2_{\Delta x}}}{dt} \leq R|q|_1^2 - \lambda_{m+1}|q|^2_{L^2_{\Delta x}} - \operatorname{Re}\left\{(1+i\mu)\sum_{j=0}^{J-1}\Delta x\overline{q_j}\mathcal{Q}_m(|U_j|^2U_j - |V_j|^2V_j)\right\}$$

followed by Lemma 5.1.2 to get

$$\frac{1}{2}\frac{d|p|^2_{L^2_{\Delta x}}}{dt} \geq R|p|^2_{L^2_{\Delta x}} - \lambda_m|p|^2_{L^2_{\Delta x}} - \operatorname{Re}\left\{(1+i\mu)\sum_{j=0}^{J-1}\Delta x\overline{p_j}\mathcal{P}_m(|U_j|^2U_j - |V_j|^2V_j)\right\} \tag{5.1.16}$$

and,

$$\begin{aligned}\frac{1}{2}\frac{d|q|^2_{L^2_{\Delta x}}}{dt} \leq\; & R|q|^2_{L^2_{\Delta x}} - (\lambda_m + \lambda_1 + 2\lambda_m^{1/2}\lambda_1^{1/2})|q|^2_{L^2_{\Delta x}} & (5.1.17)\\ & + \pi^2\Delta x^2\left(\lambda_m + \lambda_1 m^2 + (1+m^2)\lambda_m^{1/2}\lambda_1^{1/2}\right)|q|^2_{L^2_{\Delta x}} \\ & -\operatorname{Re}\left\{(1+i\mu)\sum_{j=0}^{J-1}\Delta x\overline{q_j}\mathcal{Q}_m(|U_j|^2U_j - |V_j|^2V_j)\right\}. & (5.1.18)\end{aligned}$$

We now define $\Theta(t)$ by

$$\Theta(t) := |q|^2_{L^2_{\Delta x}} - |p|^2_{L^2_{\Delta x}}$$

which we shall use to determine that the cone condition is satisfied. From (5.1.16) and (5.1.18) we have that $\Theta(t)$ satisfies

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\Theta(t) &\leq (R-\lambda_m)\Theta - \lambda_1^{1/2}\left(2\lambda_m^{1/2}+\lambda_1^{1/2}\right)|q|^2_{L^2_{\Delta x}} \\
&+ \pi^2\Delta x^2\left(\lambda_m + \lambda_1 m^2 + (1+m^2)\lambda_m^{1/2}\lambda_1^{1/2}\right)|q|^2_{L^2_{\Delta x}} \\
&- \operatorname{Re}\left\{(1+i\mu)\sum_{j=0}^{J-1}\Delta x\left(\overline{q_j}\mathcal{Q}_m(|U_j|^2U_j - |V_j|^2V_j) - \overline{p_j}\mathcal{P}_m(|U_j|^2U_j - |V_j|^2V_j)\right)\right\}.
\end{aligned} \tag{5.1.19}$$

Consider the non–linear term separately and apply Lemma 3.4.1 as in Theorem 3.5.3 and the definition of B in the statement of the Theorem to get

$$\begin{aligned}
&- \operatorname{Re}\left\{(1+i\mu)\sum_{j=0}^{J-1}\Delta x\left[(\overline{q_j}\mathcal{Q}_m(|U_j|^2U_j - |V_j|^2V_j) - \overline{p_j}\mathcal{P}_m(|U_j|^2U_j - |V_j|^2V_j)\right]\right\} \\
&\leq (1+\mu^2)^{1/2}\sum_{j=0}^{J-1}\Delta x\left||q_j|^2(|U_j|^2+|V_j|^2) + \overline{q}^2U_jV_j\right| + \left||p_j|^2(|U_j|^2+|V_j|^2)+\overline{p}_j^2U_jV_j\right| \\
&\leq 3B(1+\mu^2)^{1/2}\left(|q|^2_{L^2_{\Delta x}} + |p|^2_{L^2_{\Delta x}}\right).
\end{aligned}$$

Thus equation (5.1.19) becomes

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\Theta(t) &\leq (R-\lambda_m)\Theta - \lambda_1^{1/2}\left(2\lambda_m^{1/2}+\lambda_1^{1/2}\right)|q|^2_{L^2_{\Delta x}} \\
&+ \pi^2\Delta x^2\left(\lambda_m + \lambda_1 m^2 + (1+m^2)\lambda_m^{1/2}\lambda_1^{1/2}\right)|q|^2_{L^2_{\Delta x}} \\
&+3B(1+\mu^2)^{1/2}\left(|q|^2_{L^2_{\Delta x}} + |p|^2_{L^2_{\Delta x}}\right).
\end{aligned} \tag{5.1.20}$$

By Lemma 5.1.1 and the choice of Δx the second term on the right hand side dominates the third term on the right hand side (the error term) in the following way

$$\begin{aligned}
\pi^2\Delta x^2\left(\lambda_m + \lambda_1 m^2 + (1+m^2)\lambda_m^{1/2}\lambda_1^{1/2}\right) &\leq \pi^2\Delta x^2\left(4m^2\pi^2 + 4m^2\pi^2 + (1+m^2)4\pi^2 m\right) \\
&= 4\pi^4\Delta x^2\left(2m^2 + (1+m^2)m\right) \\
&< \pi^2/2
\end{aligned}$$

and furthermore by Lemma (5.1.1)

$$\frac{1}{2}\lambda_1 > 2\left(\pi - \frac{1}{6}\pi^3\Delta x^3\right)^2 > \pi^2/2.$$

Hence,

$$\pi^2 \Delta x^2 \left(\lambda_m - \lambda_1 m^2 - (1 + m^2) \lambda_m^{1/2} \lambda_1^{1/2} \right) < \frac{1}{2} \lambda_1.$$

Thus we find the semi–discrete version of equation (3.5.55):

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\Theta(t) \;\; &\leq \;\; (R - \lambda_m)\Theta - \lambda_1^{1/2} \left(2\lambda_m^{1/2} + \frac{1}{2}\lambda_1^{1/2} \right) |q|_{L^2}^2 \\ &\quad + 3B(1 + \mu^2)^{1/2} \left(|q|_{L^2_{\Delta x}}^2 + |p|_{L^2_{\Delta x}}^2 \right). \end{aligned} \tag{5.1.21}$$

By the choice of m we see that for $|q|_{L^2_{\Delta x}}^2 \geq |p|_{L^2_{\Delta x}}^2$ that (5.1.21) reduces to

$$\frac{1}{2}\frac{d}{dt}\Theta < (R - \lambda_m)\Theta(t) - \frac{1}{2}\lambda_1 |q|_{L^2_{\Delta x}}^2$$

and so outside the cone we have exponential decay and on the surface of the cone we see $\frac{d}{dt}\Theta < 0$ so that once the cone condition is satisfied it remains satisfied for all time.□

Note

- We see that when Δx satisfies $\Delta x < \Delta x_1$ where Δx_1 is given by (5.1.13) then we have the same form of bound on the number of Fourier modes required as in the continuous case.

5.2 Lyapunov Exponents for the Semi–Discrete System

In this section we aim to find upper and lower bounds on the sum of the first m global Lyapunov exponents defined in Definition 1.2.29 recalled below.

Recall :

Let $S(t)$ be the non–linear semigroup for the evolution equation

$$U_t = F(U),$$

in a Hilbert space X (inner product $< \bullet, \bullet > = \| \bullet \|_X^2$). Let $L(t, U^0)$ be the linearization of $S(t)$ about a solution with initial condition U^0. We assume the existence of a global attractor $\mathcal{A}$. Let $\{\xi_i(t)\}$ be an orthonormal set of eigenvectors for the operator

$$[L(t, U^0)^* L(t, U^0)]^{1/2},$$

with corresponding eigenvalues α_i. Then the ith global Lyapunov exponent μ_i is defined by

$$\mu_i := \limsup_{t \to \infty} \frac{1}{t} \log \left\{ \sup_{U^0 \in \mathcal{A}} \|L(t, U^0)\xi_i\|_X \right\}.$$

Furthermore we recall that the sum of the first m Lyapunov exponents governs the exponential growth rate of an m- volume and that from Theorem 3.5.4

$$\sum_{i=1}^{m} \mu_i = \limsup_{t \to \infty} \frac{1}{t} \log \left\{ \sup_{U^0 \in \mathcal{A}} \sup_{|\xi_i|_2^2 \leq 1} \exp \left[\mathrm{Re} \int_0^t \mathrm{Tr}\ (DF[U(s)] \cdot P(\xi_1(t), \cdots, \xi_m(t)) ds \right] \right\}. \tag{5.2.1}$$

We employ in this spatially discrete setting the techniques introduced by Constantin and Foias [27] for the $2D$ Navier–Stokes equation and extended and generalized in Constantin et al [30]. This was first applied to the continuous Ginzburg–Landau equation (3.1) by Ghidaglia and Héron [55], and a summary of their results and the results of Doering et al was presented in Section (3.5.3). Recently, using similar techniques, Yin Yan [138] has obtained bounds on the Hausdorff dimension of the global attractors for the semi–discrete approximation to the Sine–Gorden and Schrödinger equation.

Notation

- Let $\{v_i\}_{i=1}^m$ be vectors in $\mathbb{C}^J_{\text{per}}$. We define the projection $P(v_1, \cdots, v_m)$ to be the orthogonal projection

$$P(v_1, \cdots, v_m) : \mathbb{C}^J_{\text{per}} \to \operatorname{Sp}\{v_1, \cdots, v_m\}.$$

We proceed to find upper and lower bounds for the Lyapunov exponents of the semi–discrete complex Ginzburg–Landau equation (3.6.9). For which we will require the linearization of the **SD** problem. This is given in the following lemma.

Lemma 5.2.1 *The linear evolution of the semi–discrete Ginzburg–Landau equation is given for $\xi \in \mathbb{C}^J_{\text{per}}$ by*

$$\frac{d}{dt}\xi = R\xi - (1 + i\nu)M^{-1}A\xi - 2(1 + i\mu)\left(G(|U|^2)\xi - G(U^2)\overline{\xi}\right) \tag{5.2.2}$$

where we recall it is understood that for $V \in \mathbb{C}^J_{\text{per}}$,

$$|V|^2 := \left(|V_0|^2, |V_1|^2, \cdots, |V_{J-1}|^2\right)^T, V^2 := \left(V_0^2, V_1^2, \cdots, V_{J-1}^2\right)^T$$

and $G(V)$ is the diagonal matrix with entries V_j down the diagonal. Thus,

$$G(V^2) = \begin{pmatrix} V_0^2 & & 0 \\ & \ddots & \\ 0 & & V_{J-1}^2 \end{pmatrix} \quad \textit{and} \quad G(|V|^2) = \begin{pmatrix} |V_0|^2 & & 0 \\ & \ddots & \\ 0 & & |V_{J-1}|^2 \end{pmatrix}.$$

Proof Write the semi–discrete problem **SD** in component form to get for $j = 0, \cdots J - 1$

$$\frac{d}{dt}U_j = RU_j + (1 + i\nu)\delta^2 U_j - (1 + i\mu)|U_j|^2 U_j.$$

Considering $U_j + \zeta_j$ and linearize the equation for ζ to get

$$\frac{d}{dt}\zeta_j = R\zeta_j + (1 + i\nu)\delta^2\zeta_j - 2(1 + i\mu)|U_j|^2\zeta_j - (1 + i\mu)U_j^2\overline{\zeta_j}. \tag{5.2.3}$$

Writing (5.2.3) in vector notation with $\xi = (\zeta_0, \cdots, \zeta_{J-1})^T$ the lemma is proved. □

We obtain our upper and lower bounds on the Lyapunov exponents for the semi–discrete Ginzburg–Landau equation by bounding the trace term in (3.5.67).

Corollary 5.2.1

The sum of the first m–global Lyapunov exponents for (3.6.9) *satisfies :*

$$\mu_1 + \cdots + \mu_m \geq \sum_{k=-(m-1)/2}^{(m-1)/2} (R - \lambda_k) \text{ for } m \text{ odd}$$

and

$$\mu_1 + \cdots + \mu_m \geq \sum_{k=-(m/2-1)}^{m/2} (R - \lambda_k) \text{ for } m \text{ even,}$$

where λ_k is the kth eigenvalue of the matrix $M^{-1}A$ as defined in Lemma 3.7.1.

Proof To achieve this lower bound we use the fact that $U \equiv 0$ is a stationary solution and is therefore on the global attractor. Hence,

$$\begin{aligned}
\sup_{U^0} \sup_{|\xi_i|_2^2 \leq 1} \exp\left[\mathrm{Re} \int_0^t \mathrm{Tr}\ (DF[U(s)] \cdot P(\xi_1(t), \cdots, \xi(t)_m))ds\right] \\
\geq \sup_{|\xi_i|_2^2 \leq 1} \exp\left[\mathrm{Re} \int_0^t \mathrm{Tr}\ (DF[0] \cdot P(\xi_1(t), \cdots, \xi(t)_m))ds\right] \\
\geq \exp\left[\mathrm{Re} \int_0^t \mathrm{Tr}(DF[0] \cdot P(\psi_0, \psi_1, \psi_{-1} \cdots)ds\right]
\end{aligned}$$

where ψ_i are the eigenfunctions of the discrete linear operator $M^{-1}A$ and $P(\psi_0, \psi_1, \psi_{-1}, \cdots)$ is the projection onto the first m such functions. Then,

$$\mathrm{Tr}\ (DF[0]) \cdot P(\psi_0, \psi_1, \psi_{-1}, \cdots) = \sum_{k=-(m-1)/2}^{(m-1)/2} \langle \psi_k, [RI - (1 + i\nu)M^{-1}A]\psi_k \rangle \text{ for } m \text{ odd,}$$

$$\mathrm{Tr}\ (DF[0]) \cdot P(\psi_0, \psi_1, \psi_{-1}, \cdots) = \sum_{k=-(m/2-1)}^{m/2} \langle \psi_k, [RI - (1 + i\nu)M^{-1}A]\psi_k \rangle \text{ for } m \text{ even.}$$

Noting that

$$\langle \psi_k, [RI - (1 + i\nu)M^{-1}A]\psi_k \rangle = R - (1 + i\nu)\lambda_k$$

and substituting into (3.5.67) we have the desired result.□

We now establish an upper bound.

Corollary 5.2.2

The sum of the first m-global Lyapunov exponents for the semi-discrete problem (3.6.9) *satisfies :*

$$\mu_1 + \cdots + \mu_m \leq \sum_{k=-(m-1)/2}^{(m-1)/2} (R + \delta |U|^2_{L^\infty} - \lambda_k) \text{ if } m \text{ odd} \tag{5.2.4}$$

$$\mu_1 + \cdots + \mu_m \leq \sum_{k=-(m/2-1)}^{m/2} (R + \delta |U|^2_{L^\infty} - \lambda_k) \text{ if } m \text{ even} \tag{5.2.5}$$

where

$$\delta = \max\left\{0, -2 + (1+\mu^2)^{1/2}\right\}$$

and λ_k is the k^{th} eigenvalue of the matrix $M^{-1}A$.

Proof Let ϕ_k be a set of orthonormal vector spanning $P(\xi_1(t), \cdots, \xi_m(t)) L^2_{\Delta x}$ at time $t > 0$. Then,

$$\text{Re } \{\text{Tr}\,(DF[U(t)] \cdot P(\xi_1, \cdots, \xi_m))\} = \sum_{k=-(m-1)/2}^{(m-1)/2} < \phi_k, DF(t,U)\phi_k > \text{ if } m \text{ odd} \tag{5.2.6}$$

$$\text{Re } \{\text{Tr}\,(DF[U(t)] \cdot P(\xi_1, \cdots, \xi_m))\} = \sum_{k=-(m/2-1)}^{m/2} < \phi_k, DF(t,U)\phi_k > \text{ if } m \text{ even} \tag{5.2.7}$$

and for any k we have

$$\begin{aligned} < \phi_k, DF(t,U)\phi_k > = < \phi_k, (RI - M^{-1}A)\phi_k > -2 < \phi_k, G(|U|^2)\phi_k > \\ - \text{Re}\left\{(1+i\mu) < \phi_k, G(U^2)\overline{\phi_k} >\right\}. \end{aligned} \tag{5.2.8}$$

We restrict attention to the last two terms in the summation. For any $\phi \in L^2_{\Delta x}$ with U^0 on the global attractor we have:

$$\begin{aligned} -2 < \phi, G(|U|^2)\phi > &- \text{Re}\left[(1+i\mu) < \phi, G(U^2)\overline{\phi} >\right] \\ &= -2\sum_{j=0}^{J-1} \Delta x \phi_j |U_j|^2 \overline{\phi_j} - \text{Re}\left[(1+i\mu)\sum_{j=0}^{J-1} \Delta x \overline{U_j^2 \phi_j^2}\right] \\ &\leq -2\sum_{j=0}^{J-1} \Delta x |U_j|^2 |\phi|^2 + |1+i\mu| \sum_{j=0}^{J-1} \Delta x |U_j|^2 |\phi_j|^2 \\ &\leq \delta |U|^2_{L^\infty_{\Delta x}} |\phi|^2_{L^2_{\Delta x}}, \end{aligned} \tag{5.2.9}$$

where δ is defined in the statement of the theorem. Thus for any k (5.2.8) becomes

$$< \phi_k, DF(t,U)\phi_k > \leq < \phi_k, (RI - M^{-1}A)\phi_k > \delta|U|^2_{L^\infty_{\Delta x}}|\phi_k|^2_{L^2_{\Delta x}}$$

and combining this with (5.2.6), (5.2.7) and Lemma 3.5.6 we find the result. □

Remarks

▷ We note that the results above place no restriction on the spatial step Δx other than that the global attractor exists.

▷ The upper and lower bounds that were found in Corollaries 5.2.1 and 5.2.2 are of exactly the same form as the upper and lower bounds given in Corollaries 3.5.1 and 3.5.2 for the continuous equation.

▷ From the upper bound of Corollary 5.2.2 we may easily find bounds on the Lyapunov dimension of the semi–discrete global attractor $\mathcal{A}_{\Delta x}$ and hence via the Kaplan–Yorke conjecture postulate an upper bound on the Hausdorff dimension of the attractor.

▷ It would be interesting to extend the analysis to cover mappings as this would allow us to obtain bounds on the global Lyapunov exponents and Lyapunov dimension for the numerical schemes (3.6.12), (3.6.16) etc. To achieve this a discrete version of Theorem 3.5.4 is required. This will be investigated in future work.

5.3 A Fully Discrete Inertial Manifold

Notation: Employing the same notation as in Section 5.1 we let

- $\mathcal{P}_m$ denote the projection onto low Fourier modes,
- $\mathcal{Q}_m$ denote the projection onto high Fourier modes,
- $Y_{\Delta x} := \mathcal{P}_m L^2_{\Delta x}$ and $Z_{\Delta x} := \mathcal{Q}_m L^2_{\Delta x}$.

Recall the fully discrete problem **DI** (3.6.12) given by:

$$\frac{U^{n+1} - U^n}{\Delta t} + (1 + i\nu)\tilde{A}U^{n+1} = F(U^{n+1}),$$

where

$$F(V) := (R - (1 + i\nu))V - (1 + i\mu)G(|V|^2)V.$$

For the purposes of this section we assume that $U^0 \in \mathcal{B}_1(\rho_1)$, where ρ_1 was defined in Theorem 4.4.2.

We shall prove the existence of an inertial manifold for (3.6.12) inside the $H^1_{\Delta x}$ absorbing ball $\mathcal{B}_1(\rho_1)$ by proving existence of a global inertial manifold for the *prepared equation.* The prepared equation has the same dynamics as (3.6.12) for solutions in $\mathcal{B}_1(\rho_1)$ but satisfies a global Lipschitz property. The use of a prepared equation and our proof of an inertial manifold for it is an application of the analysis described in [83] and [48] which is based on mappings in Banach spaces and hence is readily applicable to this fully discrete setting.

We start by defining a suitable cut–off function which will allow us to define the prepared equation.

Let $\theta : \mathbb{R}^+ \to [0, 1]$ be a C^∞ function such that

$$\theta(s) = \begin{cases} 1 & 0 \le s \le \rho_1 \\ 0 & s \ge 2\rho_1 \end{cases} \tag{5.3.10}$$

and

$$|\theta'(s)| \le 2 \ \forall s \ge 0. \tag{5.3.11}$$

Lemma 5.3.1 *Define the function $F_\theta : \mathbb{C}^J_{\text{per}} \to \mathbb{C}$ by*

$$F_\theta(U) := \theta(\|U\|_{H^1_{\Delta x}})F(U),$$

where θ is defined by (5.3.10) *and* (5.3.11). *Then $\exists\ K_3,\ K_4 > 0$ such that $F_\theta(U)$ satisfies the estimates*

$$\begin{aligned} \|F_\theta(U)\|_{H^1_{\Delta x}} &\leq K_3 \ \forall\ U \in \mathbb{C}^J_{\text{per}} \\ \|F_\theta(U) - F_\theta(V)\|_{H^1_{\Delta x}} &\leq K_4\|U-V\|_{H^1_{\Delta x}} \ \forall\ U,V \in \mathbb{C}^J. \end{aligned}$$

Proof Note that since $F(0) = 0$ we have that $F_\theta(0) = 0$ and hence it is sufficient to prove the second inequality alone. There are three possible cases to consider.

i) If $\|U\|_{H^1_{\Delta x}}, \|V\|_{H^1_{\Delta x}} \geq 2\rho_1$ then by the definition of θ the result is obvious.

ii) If $\|U\|_{H^1_{\Delta x}} \leq 2\rho \leq \|V\|_{H^1_{\Delta x}}$ (or $\|V\|_{H^1_{\Delta x}} \leq 2\rho \leq \|U\|_{H^1_{\Delta x}}$).
Then since $\theta(\|V\|_{H^1_{\Delta x}}) = 0$ we have

$$\begin{aligned} &\left\|\theta(\|U\|_{H^1_{\Delta x}})F(U) - \theta(\|V\|_{H^1_{\Delta x}})F(V)\right\|_{H^1_{\Delta x}} \\ &\quad= \left\|\theta(\|U\|_{H^1_{\Delta x}})F(U) - \theta(\|V\|_{H^1_{\Delta x}})F(U)\right\|_{H^1_{\Delta x}} \\ &\quad\leq |\theta(\|U\|_{H^1_{\Delta x}}) - \theta(\|V\|_{H^1_{\Delta x}})|\ \|F(U)\|_{H^1_{\Delta x}} \\ &\quad\leq 2\|U-V\|_{H^1_{\Delta x}}\|F(U)\|_{H^1_{\Delta x}}. \end{aligned}$$

By Lemma 5.1.5 we have a uniform bound C on $\|F(U)\|_{H^1_{\Delta x}}$ for $U \in \mathcal{B}_1$. Hence,

$$\|F_\theta(U) - F_\theta(V)\|_{H^1_{\Delta x}} \leq K_4\|U-V\|_{H^1_{\Delta x}}.$$

iii) If $\|U\|_{H^1_{\Delta x}}, \|V\|_{H^1_{\Delta x}} \leq 2\rho_1$, then by Lemma 5.1.5 and the definition of the cut–off function θ

$$\begin{aligned} &\|F_\theta(U) - F_\theta(V)\|_{H^1_{\Delta x}} \\ &= \left\|(\theta(\|U\|_{H^1_{\Delta x}}) - \theta(\|V\|_{H^1_{\Delta x}}))F(U) + \theta(\|V\|_{H^1_{\Delta x}})\,(F(U) - F(V))\right\|_{H^1_{\Delta x}} \\ &\leq |\theta(\|U\|_{H^1_{\Delta x}}) - \theta(\|V\|_{H^1_{\Delta x}})|\ \|F(U)\|_{H^1_{\Delta x}} + |\theta(\|V\|_{H^1_{\Delta x}})|\ \|F(U) - F(V)\|_{H^1_{\Delta x}} \\ &\leq K_4\|U-V\|_{H^1_{\Delta x}}. \qquad \square \end{aligned}$$

We now define the prepared equation :

$$\frac{U^{n+1}-U^{n}}{\Delta t}+(1+i\nu)\tilde{A}U^{n+1}=F_{\theta}(U^{n+1}). \tag{5.3.12}$$

By the previous Lemma, all solutions to (5.3.12) are the same as for (3.6.12) inside the ball $\mathcal{B}_1(\rho_1)$. Hence to comprehend the long time dynamics of (3.6.12) it is sufficient to examine (5.3.12). We prove the existence of a global invariant manifold for (5.3.12) which yields the existence of an invariant manifold for (3.6.12) inside the ball.

We return momentarily to the abstract setting.

Let X be a Hilbert space with inner–product $< \bullet, \bullet >$ and let A be a densely defined sectorial operator on X. Define X^γ by $X^\gamma := D(A^\gamma)$ and endow this space with the natural norm $|\bullet|_{X^\gamma} :=< A^{\gamma/2}, A^{\gamma/2} >$. Let P_m denote the orthogonal projection onto the first m eigenfunctions of A, let $Q_m = I - P_m$ and define $Y = P_mX$ and $Z = Q_mX$ so that $X = Y \oplus Z$.

Let $G : X^\gamma \to X^\gamma$, be a mapping of the form $G(U) = LU + N(U)$, where $L, N(U) : X^\gamma \to X^\gamma$. So given $U^m \in X^\gamma$ the mapping G gives U^{m+1}:

$$U^{m+1} = G(U^m).$$

Let us make the following assumptions on G:

Assumptions G

There exist positive constants a, b, c, B such that:

$$||Lz||_{X^\gamma} \le a||z||_{X^\gamma} \qquad \forall z \in Z; \tag{G1}$$

$$\exists! w \in Y : Lw = p,\ \forall p \in Y\ \ \&\ \ b||y||_{X^\gamma} \le ||Ly||_{X^\gamma} \le c||y||_{X^\gamma} \quad \forall y \in Y; \tag{G2}$$

$$||\mathcal{R}(N(u) - N(v))||_{X^\gamma} \le B|u - v|_\gamma\ \ \forall u, v \in X^\gamma, \quad ||\mathcal{R}N(u)||_{X^\gamma} \le B\ \ \forall u \in X^\gamma, \tag{G3}$$

where $\mathcal{R}$ equals either I, P_m or Q_m.

Under these assumptions Jones and Stuart [83] prove the existence of an inertial manifold for the mapping G whenever the following conditions are fulfilled:

Conditions C

There exist constants $\delta, \epsilon \in (0, \infty)$ and $\mu \in (0,1)$ such that:

$$b^{-1}B(1+\delta) \leq \mu. \tag{C1}$$

$$a\epsilon + B \leq \epsilon. \tag{C2}$$

$$\theta := a\delta + B(1+\delta) \leq \delta\phi, \tag{C3}$$

where $\phi := b - B(1+\delta) > 0$ by (C1).

$$a + B(1+\delta) \leq \mu. \tag{C4}$$

Conditions C' There exist $\delta', \epsilon' > 0, K > 1$ and $\mu \in (0,1)$ such that Conditions (C1)–(C4) hold for all $\delta \in [\delta', K\delta'], \epsilon \in [\epsilon', K\epsilon']$.

Theorem 5.3.1

Suppose that Assumptions G and conditions C' hold for the mapping G. Then there exists a unique $\Phi : Y \to Z$ such that

$$\mathcal{M} = graph(\Phi)$$

satisfies

$$Qu^m = \Phi(P_m) \text{ if and only if } Q_m u^{m+1} = \Phi(u^{m+1})$$

and

$$dist_{X^\gamma}(u^{m+1}, \mathcal{M}) \leq C_1 \eta^m.$$

Proof See section 2 of Jones and Stuart [83] or [48].□

For the discrete problem (3.6.12) we simply find the mapping $G_{\Delta x, \Delta t}$ from (5.3.12).

Rearranging (5.3.12) we get

$$\left[I + \Delta t(1+i\nu)\widetilde{A}\right] U^{n+1} = U^n + \Delta t F_\theta(U^{n+1}),$$

and so

$$U^{n+1} = \left[I + \Delta t(1+i\nu)\widetilde{A}\right]^{-1} U^n + \Delta t \left[I + \Delta t(1+i\nu)\widetilde{A}\right]^{-1} F_\theta(U^{n+1}) \tag{5.3.13}$$

or

$$U^{n+1} = G_{\Delta x, \Delta t}(U^n). \tag{5.3.14}$$

Thus we find the mapping $G_{\Delta x,\Delta t}$ for (5.3.12) is given for $U \in \mathbb{C}^{J}_{\text{per}}$

$$G_{\Delta x,\Delta t}U := LU + N(U) \tag{5.3.15}$$

where

$$L_{\Delta x,\Delta t} = \left[I + \Delta t(1+i\nu)\widetilde{A}\right]^{-1} \tag{5.3.16}$$

and

$$N_{\Delta x,\Delta t}(U^n) := \Delta t\left[I + \Delta t(1+i\nu)\widetilde{A}\right]^{-1} F_\theta(S^1_{\Delta x}U^{n+1}). \tag{5.3.17}$$

We now show that our map $G_{\Delta x,\Delta t}$ satisfies the assumptions G, with $A = \widetilde{A}$, $G = G_{\Delta x,\Delta t}$ and with the relevant projections.

Lemma 5.3.2

For all $\Delta x < 1/(m+1)$ and $\Delta t > 0$ $\exists K_5 \in \mathbb{R}$ such that the mapping $G_{\Delta x,\Delta t}$ defined by (5.3.15) satisfies assumptions G with $\gamma = 1$ and

$$\begin{aligned}
a &:= |1+\Delta t(1+i\nu)\widetilde{\lambda}_{m+1}|^{-1} && (5.3.18)\\
b &:= |1+\Delta t(1+i\nu)\widetilde{\lambda}_{m}|^{-1} && (5.3.19)\\
c &:= |1+\Delta t(1+i\nu)\widetilde{\lambda}_{0}|^{-1} && (5.3.20)\\
B &:= \Delta t K_5|1+\Delta t(1+i\nu)\widetilde{\lambda}_{0}|^{-1}. && (5.3.21)
\end{aligned}$$

Proof First note that given any $V \in H^1_{\Delta x}$ with Fourier coefficients V_k and any $\alpha \in \mathbb{R}$ we have,

$$\|(I + \Delta t(1+\imath\nu)\widetilde{A})^\alpha V\|^2_{H^1_{\Delta x}} = \sum_{k=-\lceil J/2\rceil}^{\lfloor J/2\rfloor} \widetilde{\lambda}_k \left|1+\Delta t(1+i\nu)\widetilde{\lambda}_k\right|^{2\alpha} |V_k|^2. \tag{5.3.22}$$

G1) Let $z \in Z_{\Delta x}$ have Fourier coefficients a_k then by (5.3.22)

$$\begin{aligned}
\|L_{\Delta x,\Delta t}z\|^2_{H^1_{\Delta x}} &= \sum_{k=-\lceil J/2\rceil}^{\lfloor J/2\rfloor} \widetilde{\lambda}_k|1+\Delta t(1+i\nu)\widetilde{\lambda}_k|^{-2}|a_k|^2\\
&\leq |1+\Delta t(1+i\nu)\widetilde{\lambda}_{m+1}|^{-2}\|z\|^2_{H^1_{\Delta x}},
\end{aligned}$$

and so G1 is satisfied with a defined by (5.3.18).

G2) First note that given any $p \in Y_{\Delta x}$ $\exists! w \in Y_{\Delta x}$, $w = (I + \Delta t(1+i\nu)\widetilde{A})p$ such that $L_{\Delta x}w = p$.

Let $y \in Y_{\Delta x}$ have Fourier coefficients a_k then by (5.3.22)

$$\|L_{\Delta x,\Delta t}y\|^2_{H^1_{\Delta x}} \geq |1+\Delta t(1+i\nu)\widetilde{\lambda}_m|^{-2}\|y\|^2_{H^1_{\Delta x}},$$

and

$$\|L_{\Delta x,\Delta t}y\|^2_{H^1_{\Delta x}} \leq |1+\Delta t(1+i\nu)\widetilde{\lambda}_0|^{-2}\|y\|^2_{H^1_{\Delta x}}.$$

Thus, G2 is satisfied with b and c given by (5.3.19) and (5.3.20) respectively.

G3) Note that since $F_\theta(0) = 0$ it is sufficient to prove the second inequality alone. Let $F_\theta(S^1_{\Delta x}U^n) - F_\theta(S^1_{\Delta x}V^n)$ have Fourier coefficients b_k and consider $\|N(U^n) - N(V^n)\|_{H^1_{\Delta x}}$.

$$\begin{aligned}
\|N(U^n) - N(V^n)\|^2_{H^1_{\Delta x}} &= \Delta t\left\|\left[I+\Delta t(1+i\nu)\widetilde{A}\right]^{-1}\left(F_\theta(S^1_{\Delta x}U^n) - F_\theta(S^1_{\Delta x}V^n)\right)\right\|^2_{H^1_{\Delta x}} \\
&= \Delta t\sum_{k=-J/2\rceil}^{J/2\rfloor}\widetilde{\lambda}_k|1+(1+i\nu)\widetilde{\lambda}_k|^2|b_k|^2 \\
&\leq \Delta t|1+\Delta t(1+i\nu)\widetilde{\lambda}_0|^{-2}\ \|F_\theta(S^1_{\Delta x}U^n) - F_\theta(S^1_{\Delta x}V^n)\|^2_{H^1_{\Delta x}}
\end{aligned}$$

which, since F_θ is Lipschitz with constant K_4, becomes

$$\|N(U) - N(V)\|^2_{H^1_{\Delta x}} \leq \Delta t|1+\Delta t(1+i\nu)\widetilde{\lambda}_0|^{-2}K_4\|S^1_{\Delta x}U^n - S^1_{\Delta x}V^n\|^2_{H^1_{\Delta x}}.$$

By the continuity of the discrete semi–group $S^1_{\Delta x}$ (see section 4.4) we have that

$$\|N(U) - N(V)\|^2_{H^1_{\Delta x}} \leq \Delta t|1+\Delta t(1+i\nu)\widetilde{\lambda}_0|^{-2}K_5\|U^n - V^n\|^2_{H^1_{\Delta x}}.$$

The other inequalities may be established in a similar manner. Thus G3 is satisfied with B defined by (5.3.21). □

We now prove that conditions C' hold for **DI** (5.3.12) provided enough Fourier modes are taken.

Lemma 5.3.3 *Let $\epsilon, \delta, \alpha \in (0,1)$, $\mu \in (e^{-\alpha}, 1)$ be given and let*

$$\Delta t := \frac{\alpha}{\widetilde{\lambda}_{m+1}}.$$

Then there exists $m \in \mathbb{Z}$ and $\Delta x_0 > 0$ such that $\forall \Delta x < \Delta x_0$ the constants a, b, c, B as defined by (5.3.18–5.3.21) satisfy conditions C'.

Proof First we make our choice of $m \in \mathbb{Z}$ and $\Delta x_0 \in \mathbb{R}$ from Lemma 5.1.3 in order to satisfy the spectral gap property of Lemma 5.1.3 with

$$\begin{aligned} K_1 & \\ := & \max\left\{ \frac{\alpha(1+\delta)}{\mu} K_5 \left|1+(1+i\nu)\alpha\right|, \frac{(1+|1+(1+i\nu)\alpha|)\,|1+(1+i\nu)\alpha|K_5}{2|1+(1+i\nu)\Delta t\widetilde{\lambda}_0|\epsilon}, \right. \\ & \left. \frac{K_5\alpha(1+\delta)}{(\mu-a)|1+(1+i\nu)\Delta t\widetilde{\lambda}_0} \right\} \end{aligned} \tag{5.3.23}$$

and

$$K_2 := \frac{B(1+\delta)^2}{2\delta\Delta t} 2|1+(1+i\nu)\alpha|^3. \tag{5.3.24}$$

Before proving the conditions $C1$–$C4$ we note the following two useful equalities : $\forall x \in \mathbb{R}$,

$$|1+(1+i\nu)x|^2 = 1+2x+(1+\nu^2)x^2 \tag{5.3.25}$$

and for all $x, z \in \mathbb{R}$,

$$\begin{aligned} \frac{x-z}{\sqrt{x}+\sqrt{z}} &= \frac{(\sqrt{x}-\sqrt{z})\,(\sqrt{x}+\sqrt{z})}{\sqrt{x}+\sqrt{z}} \\ &= \sqrt{x}-\sqrt{z}\,. \end{aligned} \tag{5.3.26}$$

C1) By the choice of m and Δx we have that

$$\widetilde{\lambda}_{m+1} \geq \frac{\alpha(1+\delta)}{\mu} K_5 \left|1+(1+i\nu)\alpha\right|.$$

Using the definition of Δt, the fact that $\lambda_m < \lambda_{m+1}$ and (5.3.25) we find

$$\begin{aligned} \mu &\geq \Delta t(1+\delta)K_5\,|1+(1+i\nu)\alpha| \\ &\geq \Delta t(1+\delta)K_5 \left(1+2\alpha\frac{\widetilde{\lambda}_m}{\widetilde{\lambda}_{m+1}} + \alpha^2\frac{\widetilde{\lambda}^2_m}{\widetilde{\lambda}^2_{m+1}}(1+\nu^2)\right)^{1/2} \\ &\geq \Delta t(1+\delta)K_5 \left(1+2\Delta t\widetilde{\lambda}_m + \Delta t^2\widetilde{\lambda}^2_m(1+\nu^2)\right)^{1/2}. \end{aligned}$$

Hence

$$\mu \geq b^{-1}M(1+\delta),$$

which is exactly condition C1.

C2) By the choice of m and Δx we have

$$\widetilde{\lambda}_{m+1} \geq \frac{(1+|1+(1+i\nu)\alpha|)\,|1+(1+i\nu)\alpha|K_5}{2|1+(1+i\nu)\widetilde{\lambda}_0|\epsilon}.$$

Multiplying both sides by Δt and using the definition of Δt and B we get

$$2\epsilon\Delta t\widetilde{\lambda}_{m+1} \geq \left(1+|1+(1+i\nu)\Delta t\widetilde{\lambda}_{m+1}|\right)|1+(1+i\nu)\Delta t\widetilde{\lambda}_{m+1}|B. \tag{5.3.27}$$

If we further multiply by ϵ and add in the term $\Delta t^2\epsilon^2\widetilde{\lambda}_{q+1}$ on the left-hand side (5.3.27) implies that

$$\begin{aligned}&\epsilon^2\left(\Delta t\widetilde{\lambda}_{m+1}+\Delta t^2\widetilde{\lambda}_{m+1}\right)+\epsilon^2-\epsilon^2\\ &\geq \epsilon\left(1+|1+(1+i\nu)\widetilde{\lambda}_{m+1}|\right)\ |1+(1+i\nu)\widetilde{\lambda}_{m+1}|\ B,\end{aligned}$$

from which we see

$$\frac{\epsilon^2|1+(1+i\nu)\Delta t\widetilde{\lambda}_{m+1}|^2-\epsilon^2}{\epsilon+\epsilon|1+(1+i\nu)\Delta t\widetilde{\lambda}_{m+1}|} \geq |1+(1+i\nu)\Delta t\widetilde{\lambda}_{m+1}|B.$$

Now by the equality (5.3.26) this is equivalent to

$$\epsilon|1+(1+i\nu)\Delta t\widetilde{\lambda}_{m+1}|-\epsilon \geq |1+(1+i\nu)\Delta t\widetilde{\lambda}_{m+1}|B,$$

and so

$$\epsilon \geq B+\epsilon\frac{1}{|1+(1+i\nu)\Delta t\widetilde{\lambda}_{m+1}|},$$

Hence $\epsilon \geq B+\epsilon a$ and C2 is proved.

C3) The choice of m and Δx gives us that

$$\widetilde{\lambda}_{m+1}-\widetilde{\lambda}_m \geq \frac{B(1+\delta)^2}{2\delta\Delta t}2|1+(1+i\nu)\alpha|^3.$$

Then rearranging and using that $\widetilde{\lambda}_{m+1} > \widetilde{\lambda}_m > 0$ we find

$$\begin{aligned}2\Delta t(\widetilde{\lambda}_{m+1}-\widetilde{\lambda}_m) &\geq \frac{B(1+\delta)^2}{\delta}|1+(1+i\nu)\Delta t\widetilde{\lambda}_{m+1}||1+(1+i\nu)\Delta t\widetilde{\lambda}_m|\\ &\quad\times\left(|1+(1+i\nu)\Delta t\widetilde{\lambda}_{m+1}|+|1+(1+i\nu)\Delta t\widetilde{\lambda}_m|\right).\end{aligned}$$

This implies that

$$\begin{aligned}&2\Delta t(\widetilde{\lambda}_{m+1}-\widetilde{\lambda}_m)+\Delta t^2(1+\nu^2)(\widetilde{\lambda}_{m+1}-\widetilde{\lambda}_m)\\ &\geq \frac{B(1+\delta)^2}{\delta}|1+(1+i\nu)\Delta t\widetilde{\lambda}_{m+1}||1+(1+i\nu)\Delta t\widetilde{\lambda}_m|\\ &\quad\times\left(|1+(1+i\nu)\Delta t\widetilde{\lambda}_{m+1}|+|1+(1+i\nu)\Delta t\widetilde{\lambda}_m|\right)\end{aligned}$$

where we have added the positive term $\Delta t^2(1+\nu^2)(\widetilde{\lambda}_{m+1}-\widetilde{\lambda}_m)$. Taking terms over to the left–hand side we see that

$$\frac{\left(1+2\Delta t\widetilde{\lambda}_{m+1}+\Delta t(1+\nu^2)\right)-\left(1+\Delta t\widetilde{\lambda}_m+\Delta t^2(1+\nu^2)\widetilde{\lambda}_m\right)}{|1+(1+i\nu)\widetilde{\lambda}_{m+1}||1+(1+i\nu)\widetilde{\lambda}_m|\left(|1+(1+i\nu)\widetilde{\lambda}_{m+1}|+|1+(1+i\nu)\widetilde{\lambda}_m|\right)} \geq \frac{B(1+\delta)^2}{\delta}.$$

Now use the equality (5.3.26) to get

$$\begin{aligned} \frac{|1+(1+i\nu)\Delta t\widetilde{\lambda}_{m+1}|-|1+(1+i\nu)\Delta t\widetilde{\lambda}_m|}{|1+(1+i\nu)\Delta t\widetilde{\lambda}_{m+1}||1+(1+i\nu)\Delta t\widetilde{\lambda}_m|} &\geq \frac{B(1+\delta)^2}{\delta} \\ \frac{1}{|1+(1+i\nu)\widetilde{\lambda}_m|}-\frac{1}{|1+(1+i\nu)\widetilde{\lambda}_{m+1}|} &\geq \frac{B(1+\delta)^2}{\delta} \\ \delta(a-b) &\geq B(1+\delta)^2. \end{aligned}$$

Noting that $(1+\delta)^2=(1+\delta)+\delta(1+\delta)$ we find

$$\delta a+(1+\delta)B\leq\delta b-\delta(1+\delta)B,$$

and hence C3 is proved.

C4) The choice of m and Δx gives us from (5.3.23) that

$$\widetilde{\lambda}_{m+1}\geq\frac{K_5\alpha(1+\delta)}{(\mu-a)|1+(1+i\nu)\Delta t\widetilde{\lambda}_0|}.$$

Then by the definition of Δt and dividing through by $\widetilde{\lambda}_{m+1}$ we have

$$1\geq\frac{B(1+\delta)}{\mu-a},$$

and so

$$\mu\geq B(1+\delta)+a,$$

which is exactly C4.

Thus we have proved $C1$–$C4$ hold for the discrete problem **DI.** □

We now state the main result for this section, the existence of an inertial manifold for the scheme **DI.**

Theorem 5.3.2 *There exists $m\in\mathbb{N}$, $\Delta x_0\in\mathbb{R}^+$ such that $\forall\Delta x<\Delta x_0$ the scheme given by equation* (5.3.12) *has a global inertial manifold which may be represented as a graph $\Phi: Y_{\Delta x}\to Z_{\Delta x}$ and hence the scheme* (3.6.12) *has an inertial manifold which may be represented as a graph $\Phi: Y_{\Delta x}\to Z_{\Delta x}$ within $B_1(\rho_1)$.*

Proof Lemmas 5.3.2 and (5.3.3) ensure that all the conditions are satisfied for Theorem 5.3.1.□

5.3.1 The Cone Condition for the Fully Implicit Scheme DI

We have already shown that this condition holds for the semi–discretization provided we take the space dimension to be high enough. In this section we show that our implicit scheme **DI** will also satisfy the cone condition.

As in section 5.1.1 we define $\mathcal{P}_m$ to be the projection onto low wave numbers, and $\mathcal{Q}_m$ to be the projection onto high wave numbers.

Theorem 5.3.3 *Let $U^0, V^0 \in \mathcal{B}_1(\rho_1)$ where ρ_1 is the radius of the $H^1_{\Delta x}$ absorbing ball in Theorem 4.4.2. In addition define B by*

$$B := \max_{n>0} \left\{ |U^n|^2_{L^\infty_{\Delta x}}, |V^n|^2_{L^\infty_{\Delta x}} \right\};$$

let $\Delta t_0 > 0$ satisfy

$$\Delta t_0 < \frac{\pi}{8} \left\{ R + 2(1+\nu^2)^{1/2} B^{3/2} + (1+\nu^2) 4m^2\pi^2 \right\}^{-1}; \tag{5.3.28}$$

let m and Δx_0 be found from Lemma 5.1.4 so that

$$\lambda_m > \max \left\{ R, 9B^2(1+\mu^2)/4\lambda_1 \right\} \tag{5.3.29}$$

and let $\Delta x_1 > 0$ satisfy

$$\Delta x_1 < \left\{ \sqrt[3]{\frac{3}{\pi^2}}, \frac{1}{4\pi\sqrt{(2m^2 + (1+m^2)m)}}, \Delta x_0 \right\}. \tag{5.3.30}$$

Then $\forall \Delta t < \Delta t_0$, $\Delta x < \Delta x_1$ and $n > 0$, solutions U^n, V^n satisfy the cone condition with cone given by

$$\mathcal{C}_{m,1} = \left\{ W \in L^2_{\Delta x} : |\mathcal{Q}_m W|^2_{L^2_{\Delta x}} \leq |\mathcal{P}_m W|^2_{L^2_{\Delta x}} \right\}.$$

Proof

Let U^{n+1}, V^{n+1} be two solutions to the fully implicit scheme **DI**. Then the difference $U^n - V^n$ satisfies the equation

$$\begin{aligned} \frac{(U^{n+1} - V^{n+1}) - (U^n - V^n)}{\Delta t} &= R(U^{n+1} - V^{n+1}) - (1+i\nu) M^{-1} A (U^{n+1} - V^{n+1}) \\ &\quad - (1+i\mu) \left\{ G(|U^{n+1}|^2) U^{n+1} - G(|V^{n+1}|^2) V^{n+1} \right\}. \end{aligned}$$

The projection $p^{n+1} = \mathcal{P}_m\left(U^{n+1} - V^{n+1}\right)$ onto low wave numbers satisfies:

$$\begin{aligned}\frac{p^{n+1}-p^n}{\Delta t} &= Rp^{n+1} - (1+i\nu)M^{-1}Ap^{n+1}\\ &\quad - (1+i\mu)\mathcal{P}_m\left\{G(|U^{n+1}|^2)U^{n+1} - G(|V^{n+1}|^2)V^{n+1}\right\} \quad (5.3.31)\end{aligned}$$

and the projection $q^{n+1} = \mathcal{Q}_m\left(U^{n+1} - V^{n+1}\right)$ onto high wave numbers satisfies :

$$\begin{aligned}\frac{q^{n+1}-q^n}{\Delta t} &= Rq^{n+1} - (1+i\nu)M^{-1}Aq^{n+1}\\ &\quad - (1+i\mu)\mathcal{Q}_m\left\{G(|U^{n+1}|^2)U^{n+1} - G(|V^{n+1}|^2)V^{n+1}\right\}. \quad (5.3.32)\end{aligned}$$

Take the inner product of (5.3.31) with p^{n+1} and (5.3.32) with q^{n+1} to get

$$\begin{aligned}\left\langle\frac{p^{n+1}-p^n}{\Delta t}, p^{n+1}\right\rangle &= R|p^{n+1}|^2_{L^2_{\Delta x}} - (1+i\nu) < M^{-1}Ap^{n+1}, p^{n+1} >\\ &\quad - (1+i\mu)\left\langle\mathcal{P}_m\left\{G(|U^{n+1}|^2)U^{n+1} - G(|V^{n+1}|^2)V^{n+1}\right\}, p^{n+1}\right\rangle\end{aligned}$$

(5.3.33)

and

$$\begin{aligned}\left\langle\frac{q^{n+1}-q^n}{\Delta t}, q^{n+1}\right\rangle &= R|q^{n+1}|^2_{L^2_{\Delta x}} - (1+i\nu)\|q^{n+1}\|^2\\ &\quad - (1+i\mu)\left\langle\mathcal{Q}_m\left\{G(|U^{n+1}|^2)U^{n+1} - G(|V^{n+1}|^2)V^{n+1}\right\}, q^{n+1}\right\rangle.\end{aligned}$$

(5.3.34)

Let us consider equations (5.3.33) and (5.3.34) separately.

- First we consider (5.3.33) for the low wave numbers. Noting that $\forall a, b \in \mathbb{C}$

$$\operatorname{Re}\{< a-b, a >\} = |a|^2 - < b, a > - < a, b > = \frac{|a|^2 - |b|^2}{2} + \frac{|a-b|^2}{2};$$

take the real part of (5.3.33) to get

$$\begin{aligned}&\frac{|p^{n+1}|^2_{L^2_{\Delta x}} - |p^n|^2_{L^2_{\Delta x}}}{2\Delta t} + \frac{|p^{n+1}-p^n|^2_{L^2_{\Delta x}}}{2\Delta t} = R|p^{n+1}|^2_{L^2_{\Delta x}} - \|p^{n+1}\|_1^2\\ &\quad - \operatorname{Re}\left\{(1+i\mu)\left\langle\mathcal{P}_m\left\{G(|U^{n+1}|^2)U^{n+1} - G(|V^{n+1}|^2)V^{n+1}\right\}, p^{n+1}\right\rangle\right\}. \quad (5.3.35)\end{aligned}$$

We seek a bound on $|p^{n+1}-p^n|^2_{L^2_{\Delta x}}$, to which end we take the $L^2_{\Delta x}$ norm of equation (5.3.31) to find

$$\begin{aligned}\frac{|p^{n+1}-p^n|_{L^2_{\Delta x}}}{\Delta t} &\leq R|p^{n+1}|_{L^2_{\Delta x}} + (1+\nu^2)^{1/2}\|p^{n+1}\|_1\\ &\quad + (1+\mu^2)^{1/2}\left|G(|U^{n+1}|^2)U^{n+1} - G(|V^{n+1}|^2)V^{n+1}\right|_{L^2_{\Delta x}} |p^{n+1}|_{L^2_{\Delta x}}\\ &\leq R|p^{n+1}|_{L^2_{\Delta x}} + (1+\nu^2)^{1/2}\|p^{n+1}\|_1 + 2(1+\mu^2)^{1/2}B^{3/2}|p^{n+1}|_{L^2_{\Delta x}}.\end{aligned}$$

By squaring and using the simple inequality $\forall a, b \in \mathbb{R},\ (a+b)^2 \leq 2a^2 + 2b^2$ we find

$$|p^{n+1} - p^n|_{L^2_{\Delta x}} \leq 2\Delta t^2 (R + 2(1+\mu^2)^{1/2} B^{3/2})^2 |p^{n+1}|^2_{L^2_{\Delta x}} + 2\Delta t^2 (1+\nu^2) \|p^{n+1}\|_1^2.$$

We may bound $\|p^{n+1}\|_1$ by Lemma 5.1.6 to get

$$|p^{n+1} - p^n|_{L^2_{\Delta x}} \leq 2\Delta t^2 \left\{ (R + 2(1+\mu^2)^{1/2} B^{3/2})^2 + (1+\nu^2)\lambda_m \right\} |p^{n+1}|^2_{L^2_{\Delta x}}.$$

If we substitute this into (5.3.35)

$$\begin{aligned}
\frac{|p^{n+1}|^2_{L^2_{\Delta x}} - |p^n|^2_{L^2_{\Delta x}}}{2\Delta t} \quad \geq \quad & R|p^{n+1}|^2_{L^2_{\Delta x}} - \|p^{n+1}\|_1^2 \\
& -\mathrm{Re}\left\{ (1+i\mu) \left\langle \mathcal{P}_m \left\{ G(|U^{n+1}|^2) U^{n+1} - G(|V^{n+1}|^2) V^{n+1} \right\}, p^{n+1} \right\rangle \right\} \\
& - 2\Delta t \left\{ \left(R + 2(1+\mu^2)^{1/2} B^{3/2} \right)^2 + (1+\nu^2)\lambda_m \right\} |p^{n+1}|^2_{L^2_{\Delta x}},
\end{aligned}$$

expand the innerproduct and again use Lemma 5.1.6 we get

$$\begin{aligned}
\frac{|p^{n+1}|^2_{L^2_{\Delta x}} - |p^n|^2_{L^2_{\Delta x}}}{2\Delta t} \quad \geq \quad & R|p^{n+1}|^2_{L^2_{\Delta x}} - \lambda_m |p^{n+1}|^2_{L^2_{\Delta x}} \\
& -\mathrm{Re}\left\{ (1+i\mu) \sum_{j=0}^{J-1} \Delta x \overline{p}_j^{n+1} \mathcal{P}_m \left(|U_j^{n+1}|^2 U_j^{n+1} - |V_j^{n+1}|^2 V_j^{n+1} \right) \right\} \\
& - 2\Delta t \left\{ \left(R + 2(1+\mu^2)^{1/2} B^{3/2} \right)^2 + (1+\nu^2)\lambda_k \right\} |p^{n+1}|^2_{L^2_{\Delta x}}.
\end{aligned}$$

• Secondly consider equation (5.3.34) for the high wave numbers and take the real part

$$\begin{aligned}
\frac{|q^{n+1}|^2_{L^2_{\Delta x}} - |q^n|^2_{L^2_{\Delta x}}}{2\Delta t} \quad \leq \quad & R|q^{n+1}|^2_{L^2_{\Delta x}} - \|q^{n+1}\|_1^2 \\
& - \mathrm{Re}\ \left\{ (1+i\mu) \left\langle \mathcal{Q}_m \left\{ G(|U^{n+1}|^2) U^{n+1} - G(|V^{n+1}|^2) V^{n+1} \right\}, q^{n+1} \right\rangle \right\}.
\end{aligned} \tag{5.3.36}$$

Applying Lemma 5.1.6 to (5.3.36)

$$\begin{aligned}
\frac{|q^{n+1}|^2_{L^2_{\Delta x}} - |q^n|^2_{L^2_{\Delta x}}}{2\Delta t} \quad \leq \quad & R|q^{n+1}|^2_{L^2_{\Delta x}} - \lambda_{k+1} |q^{n+1}|^2_{L^2_{\Delta x}} \\
& - \mathrm{Re}\ \left\{ (1+i\mu) \left\langle \mathcal{Q}_m \left\{ G(|U^{n+1}|^2) U^{n+1} - G(|V^{n+1}|^2) V^{n+1} \right\}, q^{n+1} \right\rangle \right\},
\end{aligned}$$

followed by Lemma (5.1.2) we get

$$\frac{|q^{n+1}|^2_{L^2_{\Delta x}} - |q^n|^2_{L^2_{\Delta x}}}{2\Delta t} \quad \leq \quad R|q^{n+1}|^2_{L^2_{\Delta x}} - (\lambda_m + 2\lambda_m^{1/2}\lambda_1^{1/2} + \lambda_1)|q^{n+1}|^2_{L^2_{\Delta x}}$$

$$+\pi^2\Delta x^2\left(\lambda_m+\lambda_1 m+(1+m^2)\lambda_m^{1/2}\lambda_1^{1/2}\right)|q^{n+1}|^2_{L^2_{\Delta x}}$$

$$-\operatorname{Re}\left\{(1+i\mu)\sum_{j=0}^{J-1}\Delta x\overline{q_j}^{n+1}\mathcal{Q}_m\left(|U_j^{n+1}|^2U_j^{n+1}-|V_j^{n+1}|^2V_j^{n+1}\right)\right\}. \tag{5.3.37}$$

Now define Θ^n by

$$\Theta^n := |q^n|^2_{L^2_{\Delta x}}-|p^n|^2_{L^2_{\Delta x}}$$

which we shall use to show that the cone condition is satisfied. From (5.3.36) and (5.3.37) Θ^n satisfies the inequality

$$\begin{aligned}\frac{1}{2}\frac{\Theta^{n+1}-\Theta^n}{\Delta t}&\le(R-\lambda_m)\Theta^{n+1}-\lambda_1^{1/2}(2\lambda_m^{1/2}\lambda_1^{1/2}+\lambda_1^{1/2})|q^{n+1}|^2_{L^2_{\Delta x}}\\
&+\pi^2\Delta x^2\left(\lambda_m+\lambda_1 m+(1+m^2)\lambda_m^{1/2}\lambda_1^{1/2}\right)|q^{n+1}|^2_{L^2_{\Delta x}}\\
&-\operatorname{Re}\left\{(1+i\mu)\sum_{j=0}^{J-1}\Delta x\left(\overline{q_j^{n+1}}\mathcal{Q}_m(|U_j^{n+1}|^2U_j^{n+1}-|V_j^{n+1}|^2V_j^{n+1})\right.\right.\\
&\qquad\left.\left.-\overline{p_j^{n+1}}\mathcal{P}_m(|U_j^{n+1}|^2U_j^{n+1}-|V_j^{n+1}|^2V_j^{n+1})\right)\right\}\\
&+2\Delta t\left\{\left(R+(1+\mu^2)^{1/2}B^{3/2}\right)^2+(1+\nu^2)\lambda_m\right\}|p^{n+1}|^2_{L^2_{\Delta x}}.\end{aligned} \tag{5.3.38}$$

Now consider the non-linear term separately. We apply Lemma 3.4.1 and use the definition of B in the statement of the theorem to get

$$\begin{aligned}&-\operatorname{Re}\left\{(1+i\mu)\sum_{j=0}^{J-1}\Delta x\left(\overline{q_j}^{n+1}\mathcal{Q}_m(|U_j^{n+1}|^2U_j^{n+1}-|V_j^{n+1}|^2V_j^{n+1})\right.\right.\\
&\qquad\qquad\left.\left.-\overline{p_j^{n+1}}\mathcal{P}_m(|U_j^{n+1}|^2U_j^{n+1}-|V_j^{n+1}|^2V_j^{n+1})\right)\right\}\\
\le\;&(1+\mu^2)^{1/2}\sum_{j=0}^{J-1}\Delta x\left|\overline{q_j^{n+1}}q_j^{n+1}(|U_j^{n+1}|^2+|V_j^{n+1}|^2)+\overline{q_j^{n+1}}^2U_j^{n+1}V_j^{n+1}\right|\\
&+(1+\mu^2)^{1/2}\sum_{j=0}^{J-1}\Delta x\left|\overline{p_j^{n+1}}p_j^{n+1}(|U_j^{n+1}|^2+|V_j^{n+1}|^2)+\overline{p_j}^{n+1^2}U_j^{n+1}V_j^{n+1}\right|\\
\le\;&3B(1+\mu^2)^{1/2}\left(|q^{n+1}|^2_{L^2_{\Delta x}}+|p^{n+1}|^2_{L^2_{\Delta x}}\right).\end{aligned}$$

Thus (5.3.38) becomes

$$\begin{aligned}\frac{1}{2}\frac{\Theta^{n+1}-\Theta^n}{\Delta t}&\le(R-\lambda_m)\Theta^{n+1}-\lambda_1^{1/2}(2\lambda_m^{1/2}\lambda_1^{1/2}+\lambda_1^{1/2})|q^{n+1}|^2_{L^2_{\Delta x}}\\
&+\pi^2\Delta x^2\left(\lambda_m+\lambda_1 m+(1+m^2)\lambda_m^{1/2}\lambda_1^{1/2}\right)|q^{n+1}|^2_{L^2_{\Delta x}}\end{aligned}$$

$$+3B(1+\mu^2)^{1/2}(|q^{n+1}|^2_{L^2_{\Delta x}} + |p^{n+1}|^2_{L^2_{\Delta x}})$$
$$+ 2\Delta t\left\{\left(R + (1+\mu^2)^{1/2}B^{3/2}\right)^2 + (1+\nu^2)\lambda_m\right\}|p^{n+1}|^2_{L^2_{\Delta x}}. \qquad (5.3.39)$$

We now wish to use the second term on the right to control the last three terms when $|p^{n+1}|^2_{L^2_{\Delta x}} < |q^{n+1}|^2_{L^2_{\Delta x}}$. This is where we call upon our choices of Δx and Δt.

Notice that

$$\begin{aligned}\pi^2\Delta x^2(\lambda_m + \lambda_1 m^2 + (1+m^2)\lambda_m^{1/2}\lambda_1^{1/2}) + 2\Delta t\left\{\left(R + (1+\mu^2)^{1/2}B^{3/2}\right)^2 + (1+\nu^2)\lambda_m\right\} \\ \leq \ \pi^2\Delta x^2(4m^2\pi^2 + 4m^2\pi^2 + (1+m^2)4\pi^2 m) \\ + 2\Delta t\left\{\left(R + (1+\mu^2)^{1/2}B^{3/2}\right)^2 + (1+\nu^2)4\pi^2 m^2\right\}.\end{aligned}$$

By the choice of Δx and Δt this is easily seen to become

$$\begin{aligned}\pi^2\Delta x^2(\lambda_m + \lambda_1 m^2 + (1+m^2)\lambda_m^{1/2}\lambda_1^{1/2}) + 2\Delta t\left\{\left(R + (1+\mu^2)^{1/2}B^{3/2}\right)^2 + (1+\nu^2)\lambda_m\right\} \\ \leq \ \pi^2/4 + \pi^2/4 = \pi^2/2.\end{aligned}$$

Further note that by Lemma (5.1.1) and the choice of Δx

$$\frac{1}{2}\lambda_1 > 2\left(\pi - \frac{1}{6}\pi^3\Delta x^3\right)^2 > \pi^2/2.$$

Hence,

$$\begin{aligned}\pi^2\Delta x^2\left(\lambda_m - \lambda_1 m^2 - (1+m^2)\lambda_m^{1/2}\lambda_1^{1/2}\right) + 2\Delta t\left\{\left(R + (1+\mu^2)^{1/2}B^{3/2}\right)^2 + (1+\nu^2)\lambda_m\right\} \\ < \frac{1}{2}\lambda_1.\end{aligned}$$

Thus when $|q^{n+1}|^2_{L^2_{\Delta x}} > |p^{n+1}|^2_{L^2_{\Delta x}}$ we are outside the cone and we have that

$$\frac{\Theta^{n+1} - \Theta^n}{2\Delta t} \leq (R - \lambda_m)\Theta^{n+1} - \frac{1}{2}\lambda_1|q^{n+1}|^2_{L^2_{\Delta x}}$$

and so

$$\Theta^{n+1} \leq \left[1 - 2R\Delta t\,(R - \lambda_m)\right]^{-(n+1)}\Theta^0;$$

from which we easily see that

$$\Theta^n \to 0 \ \text{ as } n \to \infty.$$

Furthermore if $\Theta^{n+1} = 0$ then

$$\Theta^n > \frac{1}{\Delta t}\lambda_1|q^{n+1}|^2_{L^2_{\Delta x}},$$

and hence once the cone condition is satisfied it remains satisfied.□

Chapter 6

Rotating Waves

In this chapter we examine exact solutions to the semi–discrete problem **SD** (3.6.9) and the fully discrete problems **DE** (3.6.15), **DI** (3.6.12) and **DEI** (3.6.16). These solutions are discrete analogues of the rotating wave solutions discussed in section 3.5.4 for the continuous complex Ginzburg–Landau equation **CGL** (3.1).

Existence of these rotating wave solutions is proved and stability of the waves to linear perturbations is examined. The curves of neutral stability to linear perturbations are found and investigated numerically assisted by the numerical continuation program **Pitcon** [107]. Spurious periodic solutions which exist for all $\Delta t > 0$ are found in the fully discrete schemes **DE** (3.6.15), **DI** (3.6.12) and **DEI** (3.6.16).

We recall from Section 3.5.4 that the rotating wave solutions arose from Hopf–like bifurcations of the $U \equiv 0$ solution. These were not true Hopf bifurcations because of the multiple eigenvalues and it was possible that other solutions bifurcated at these bifurcation points. These were Hopf–like because the necessary conditions were satisfied and the bifurcations did give rise to periodic solutions - the rotating waves - for which we have an explicit expression.

Notation

For the purposes of this chapter we employ the following notation.
For $m, \ell \in \mathbb{Z}$, we define

$$k_m := 2m\pi; \tag{6.0.1}$$

$$S_{m+\ell} := \frac{2}{\Delta x}\sin\left(\frac{(k_m+k_l)\Delta x}{2}\right); \tag{6.0.2}$$

and

$$S_{m-\ell} := \frac{2}{\Delta x}\sin\left(\frac{(k_m-k_l)\Delta x}{2}\right). \tag{6.0.3}$$

6.1 Rotating Wave Solutions of the Semi–Discrete Problem.

In this section we show that the semi–discrete problem (3.6.9) has rotating wave solutions and we examine their stability. This is analogous to the continuous analysis outlined in section (3.5.4), the details of which may be found in [41].

Lemma 6.1.1 *The semi-discrete problem* (3.6.9) *possesses rotating wave solutions*

$$U_m := (U_{m,0},\cdots,U_{m,J-1})^T,$$

where

$$U_{m,j} := a_m e^{i(k_m j\Delta x-\omega_m t)} \tag{6.1.1}$$

and the amplitude a_m and period ω_m satisfy

$$|a_m|^2 = R-\lambda_m, \tag{6.1.2}$$

$$\omega_m = \mu R+(\nu-\mu)\lambda_m, \tag{6.1.3}$$

and

$$\lambda_m = \frac{4}{\Delta x^2}\sin^2(m\pi\Delta x),\ \ m=0,\pm 1,\cdots$$

are the eigenvalues of $M^{-1}A$ as given in Lemma 3.7.1.

Proof Recall the semi–discrete formulation (3.6.9) in component form

$$\frac{d}{dt}U_j = RU_j-(1+i\nu)\delta^2U_j-(1+i\mu)|U_j|^2U_j, \tag{6.1.4}$$

where δ^2 was defined in (3.6.2). If we substitute (6.1.1) into (6.1.4) and compare coefficients we get

$$-i\omega_m = R+(1+i\nu)\frac{\exp(-ik_m\Delta x)-2+\exp(ik_m\Delta x)}{\Delta x^2}-(1+i\mu)|a_m|^2.$$

Thus,

$$-i\omega_m = R - (1 + i\nu)\lambda_m - (1 + i\mu)|a_m|^2$$

and equating real and imaginary parts we find (6.1.2) and (6.1.3) for the amplitude and period respectively. □

Notes

1. For $m = 0$ we have

$$U_m = \left(a_m e^{-i\omega_m \Delta t}, \cdots, a_m e^{-i\omega_m \Delta t}\right)^T = a_m e^{-i\omega_m \Delta t} (1, \cdots, 1)^T$$

   which is the spatially homogeneous rotating wave, just as in the continuous case.

2. From equation (6.1.2) we see that the mth rotating wave exists for $R \geq \lambda_m$, and comes into existence at $R = \lambda_m$. This is the exact analogy of the continuous case of Section 3.5.4. As in the continuous case the rotating waves originate from Hopf–like bifurcations of the trivial solution (see Lemma (6.1.2) below). Figure 6.1 shows the bifurcation points in the (R, m) plane for $\Delta x = 1/32$, $\Delta x = 1/64$ and $\Delta x = 1/128$. In Figure 6.2 we have plotted the bifurcation diagram for $R \geq 0$ with $\Delta x = 1/64$ and $\nu = \mu = -\sqrt{3}$. The mth line from the origin is the mth rotating wave originating from the bifurcation point on the $a_m = 0$ axis (ie from the zero solution).

3. The semi discrete rotating waves of Lemma 6.1.1 converge with order $O(\Delta x^2)$ to the continuous rotating waves of Section 3.5.4 as Δx tends to zero. This is easily seen, since by an application of Taylor's theorem the eigenvalues λ_m for $M^{-1}A$ converge to the eigenvalues Λ_m of A_0 (see section 3.6).

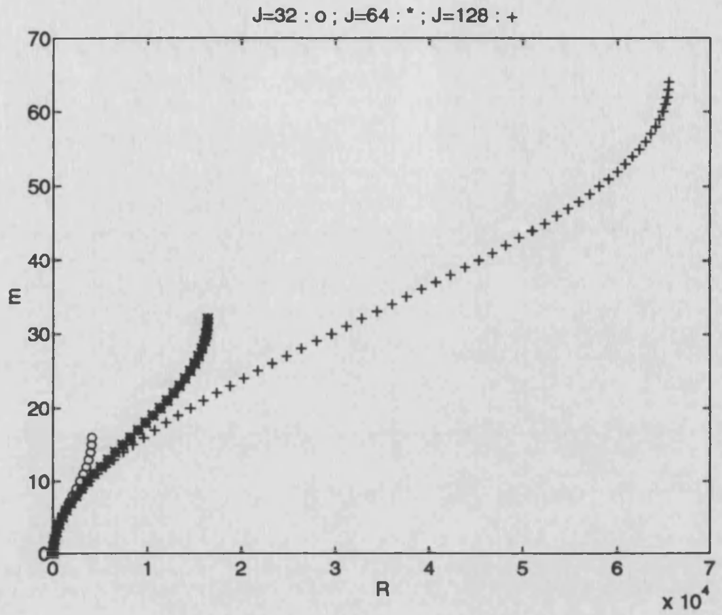

Figure 6.1: Bifurcation points for $\Delta x = 1/32 : \circ$, $\Delta x = 1/64 : \bullet$, $\Delta x = 1/128 : +$.

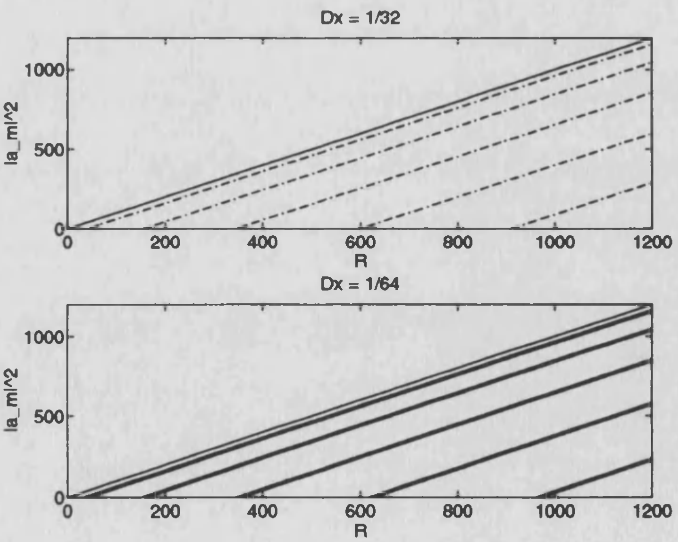

Figure 6.2: Bifurcation diagram for $R > 0$ for **SD**: $\Delta x = 1/32$ and $\Delta x = 1/64$.

Lemma 6.1.2 *The mth rotating wave U^m for (3.6.9) originates from the mth Hopf-like bifurcation point for the trivial solution $U \equiv 0$ for (3.6.9).*

Proof Linearize the semi-discrete problem **SD** (3.6.9) about a solution U to get the linear evolution equation for $\epsilon \in \mathbb{C}$

$$\epsilon_t = R\epsilon - (1+i\nu)M^{-1}A\epsilon - (1+i\mu)\left\{2G(|U|^2)\epsilon + G(U^2)\bar{\epsilon}\right\};$$

where we recall that for $V \in \mathbb{C}^J_{\text{per}}$, $G(V)$ is defined to be the $J \times J$ diagonal matrix with entries V_j, and $G(|V|^2)$ is taken as the diagonal matrix with entries $|V_j|^2$. Thus linearizing about $U \equiv 0$ we get

$$\epsilon_t = \left(RI - (1+i\nu)M^{-1}A\right)\epsilon.$$

The operator $(RI - (1+i\nu)M^{-1}A)$ has eigenvalues

$$R - (1+i\nu)\lambda_m,$$

which are imaginary if and only if $R = \lambda_m$.□

Having shown that these rotating waves exist we turn attention to their stability.

Lemma 6.1.3 *Let U_m be the mth rotating wave for (3.6.9).*
Let $\epsilon_\ell(t) = (\epsilon_{\ell,0}(t), \cdots, \epsilon_{\ell,J-1}(t))^T$ be an arbitrary perturbation to U_m. Then the linearized evolution of the jth component $\epsilon_{\ell,j}$ of the perturbation ϵ_ℓ is given by

$$\begin{aligned}\frac{d\epsilon_{\ell,j}}{dt} &= \frac{(1+i\nu)}{\Delta x^2}\left\{e^{-ik_m\Delta x}\epsilon_{\ell,j-1} - 2\epsilon_{\ell,j} + e^{ik_m\Delta x}\epsilon_{\ell,j+1}\right\} - (1+i\mu)|a_m|^2(\epsilon_{\ell,j} + \overline{\epsilon_{\ell,j}}) \\ &\quad + \epsilon_{\ell,j}(1+i\nu)\lambda_m. \qquad (6.1.5)\end{aligned}$$

Proof Substitute $U_{m,j}(1+\epsilon_{\ell,j})$ into (6.1.4) and simplify to get

$$\begin{aligned}\frac{d\epsilon_{\ell,j}}{dt} &= \frac{(1+i\nu)}{\Delta x^2}\left\{e^{-ik_m\Delta x} - 2 + e^{+ik_m\Delta x} + e^{ik_m\Delta x}\epsilon_{\ell_{j-1}} - 2\epsilon_{\ell,j} + e^{ik_m\Delta x}\epsilon_{\ell j+1}\right\} \\ &\quad + R(1+\epsilon_{\ell,j}) - (1+i\mu)(1+\epsilon_{\ell,j})|a_m|^2|1+\epsilon_{\ell,j}|^2 - i\omega_m(1+\epsilon_{\ell,j}).\end{aligned}$$

Thus,

$$\begin{aligned}\frac{d\epsilon_{\ell,j}}{dt} &= -(1+i\nu)\lambda_m + \frac{(1+i\nu)}{\Delta x^2}\left\{e^{-ik_m\Delta x}\epsilon_{\ell_{j-1}} - 2\epsilon_{\ell,j} + e^{ik_m\Delta x}\epsilon_{\ell_{j+1}}\right\} \\ &\quad + R(1+\epsilon_{\ell,j}) - (1+i\mu)(1+\epsilon_{\ell,j})|a_m|^2|1+\epsilon_{\ell,j}|^2 + i\omega_m(1+\epsilon_{\ell,j}).\end{aligned}$$

Substitute from (6.1.3) for the period ω_m

$$\begin{aligned}
\frac{d\epsilon_{\ell,j}}{dt} &= -(1+i\nu)\lambda_m + \frac{(1+i\nu)}{\Delta x^2}\left\{e^{-ik_m\Delta x}\epsilon_{\ell_{j-1}} - 2\epsilon_{\ell,j} + e^{ik_m\Delta x}\epsilon_{\ell_{j+1}}\right\} \\
&\quad + R(1+\epsilon_{\ell,j}) - (1+i\mu)(1+\epsilon_{\ell,j})|a_m|^2|1+\epsilon_{\ell,j}|^2 + i(\mu R + (\nu-\mu)\lambda_m)(1+\epsilon_{\ell,j}) \\
&= -(1+i\nu)\lambda_m + \frac{(1+i\nu)}{\Delta x^2}\left\{e^{-ik_m\Delta x}\epsilon_{\ell_{j-1}} - 2\epsilon_{\ell,j} + e^{ik_m\Delta x}\epsilon_{\ell_{j+1}}\right\} \\
&\quad + (1+\epsilon_{\ell,j})\left\{R - (1+i\mu)|a_m|^2|1+\epsilon_{\ell,j}|^2 + i(\mu R + (\nu-\mu)\lambda_m)\right\}, \qquad (6.1.6)
\end{aligned}$$

and now use the expression for the amplitude (6.1.2) to eliminate R :

$$\begin{aligned}
\frac{d\epsilon_{\ell,j}}{dt} &= -(1+i\nu)\lambda_m + \frac{(1+i\nu)}{\Delta x^2}\left\{e^{-ik_m\Delta x}\epsilon_{\ell_{j-1}} - 2\epsilon_{\ell,j} + e^{ik_m\Delta x}\epsilon_{\ell_{j+1}}\right\} \\
&\quad + (1+\epsilon_{\ell,j})\left\{|a_m|^2 + \lambda_m - (1+i\mu)|a_m|^2|1+\epsilon_{\ell,j}|^2\right. \\
&\qquad \left. + i(\mu(|a_m|^2+\lambda_m) + (\nu-\mu)\lambda_m)\right\} \\
&= -(1+i\nu)\lambda_m + \frac{(1+i\nu)}{\Delta x^2}\left\{e^{-ik_m\Delta x}\epsilon_{\ell_{j-1}} - 2\epsilon_{\ell,j} + e^{ik_m\Delta x}\epsilon_{\ell_{j+1}}\right\} \\
&\quad + (1+\epsilon_{\ell,j})(1+i\mu)|a_m|^2\left\{1 - |1+\epsilon_{\ell,j}|^2\right\} + (1+\epsilon_{\ell,j})\lambda_m\left\{1 + i\mu + i(\nu-\mu))\right\} \\
&= \frac{(1+i\nu)}{\Delta x^2}\left\{e^{-ik_m\Delta x}\epsilon_{\ell_{j-1}} - 2\epsilon_{\ell,j} + e^{ik_m\Delta x}\epsilon_{\ell_{j+1}}\right\} \\
&\quad + (1+\epsilon_{\ell,j})(1+i\mu)|a_m|^2\left\{1 - |1+\epsilon_{\ell,j}|^2\right\} + \lambda_m(1+i\nu)\epsilon_{\ell,j}. \qquad (6.1.7)
\end{aligned}$$

Neglecting non–linear terms in $\epsilon_{\ell,j}$ in equation (6.1.7) we find (6.1.5). □

We now wish to examine the stability of the mth rotating wave to linear perturbations in the ℓth mode. To achieve this we set

$$\epsilon_{\ell,j}(t) = \alpha_\ell(t)\exp(ik_\ell j\Delta x) + \alpha_{-\ell}(t)\exp(-ik_\ell j\Delta x) \qquad (6.1.8)$$

for $j = 0, \cdots, J-1$.

Lemma 6.1.4 *Consider a perturbation ϵ_ℓ with components $\epsilon_{\ell,j}$ given by* (6.1.8) *to the mth rotating wave U_m with $m \neq \ell$. Then the linear evolution equations for the amplitudes α_ℓ and $\alpha_{-\ell}$ satisfy*

$$\frac{d}{dt}\begin{pmatrix} \overline{\alpha_\ell} \\ \alpha_{-\ell} \end{pmatrix} = \begin{pmatrix} \overline{-C_+} & -(1-i\mu)|a_m|^2 \\ -(1+i\mu)|a_m|^2 & -C_- \end{pmatrix}\begin{pmatrix} \overline{\alpha_\ell} \\ \alpha_{-\ell} \end{pmatrix}, \qquad (6.1.9)$$

where

$$C_{+/-} = (1+i\nu)S^2_{m+\ell/m-\ell} + (1+i\mu)|a_m|^2 - (1+i\nu)\lambda_m. \qquad (6.1.10)$$

Proof

Substituting (6.1.8) into the linear evolution equation given by (6.1.5), yields

$$\begin{aligned}
&\frac{d\alpha_\ell}{dt}e^{ik_\ell j\Delta x}+\frac{d\alpha_{-\ell}}{dt}e^{-ik_\ell j\Delta x}\\
&= \frac{(1+i\nu)}{\Delta x^2}\left\{e^{-ik_m\Delta x}\left(\alpha_\ell e^{ik_\ell(j-1)\Delta x}+\alpha_{-\ell}e^{-ik_\ell(j-1)\Delta x}\right)\right.\\
&\quad \left.-2\left(\alpha_\ell e^{ik_\ell j\Delta x}+\alpha_{-\ell}e^{-ik_\ell j\Delta x}\right)+e^{ik_m\Delta x}\left(\alpha_\ell e^{ik_\ell(j+1)\Delta x}+\alpha_{-\ell}e^{-ik_\ell(j+1)\Delta x}\right)\right\}\\
&\quad -(1+i\mu)|a_m|^2\left(\alpha_\ell e^{ik_\ell j\Delta x}+\alpha_{-\ell}e^{-ik_\ell j\Delta x}+\overline{\alpha_\ell}e^{-ik_\ell j\Delta x}+\overline{\alpha_{-\ell}}e^{ik_\ell j\Delta x}\right)\\
&\quad +(1+i\nu)\lambda_m\left(\alpha_\ell e^{ik_\ell j\Delta x}+\alpha_{-\ell}e^{-ik_\ell j\Delta x}\right).
\end{aligned}$$

By comparing coefficients of $e^{ik_\ell j\Delta x}$ and $e^{-ik_\ell j\Delta x}$ we see that

$$\frac{d\alpha_\ell}{dt}=-\alpha_\ell(1+i\nu)S^2_{m+\ell}-(1+i\mu)|a_m|^2(\alpha_\ell+\overline{\alpha_{-\ell}})+\alpha_\ell(1+i\nu)\lambda_m \qquad (6.1.11)$$

and

$$\frac{d\alpha_{-\ell}}{dt}=-\alpha_{-\ell}(1+i\nu)S^2_{m-\ell}-(1+i\mu)|a_m|^2(\alpha_{-\ell}+\overline{\alpha_\ell})+\alpha_{-\ell}(1+i\nu)\lambda_m. \qquad (6.1.12)$$

All that remains is to take the complex conjugate of (6.1.11) and put that and (6.1.12) into the matrix formulation. □

Lemma 6.1.5 *With C_+ and C_- defined by equation* (6.1.10), *the following equalities hold:*

$$\overline{C_+}+C_-=S^2_{m+\ell}+S^2_{m-\ell}+2(|a_m|^2-\lambda_m)+i\nu(S^2_{m-\ell}-S^2_{m+\ell}), \qquad (6.1.13)$$

and

$$\begin{aligned}
\overline{C_+}C_- &= (S_{m+\ell}+|a_m|^2-\lambda_m)(S^2_{m-\ell}+|a_m|^2-\lambda_m)\\
&\quad +(\nu S^2_{m+\ell}+\mu|a_m|^2-\nu\lambda_m)(\nu S^2_{m-\ell}+\mu|a_m|^2-\nu\lambda_m)\\
&\quad +i\left\{(S^2_{m+\ell}+|a_m|^2-\lambda_m)(\nu S^2_{m-\ell}+\mu|a_m|^2-\nu\lambda_m)\right.\\
&\qquad \left.-(\nu S^2_{m+\ell}+\mu|a_m|^2-\nu S)(S^2_{m-\ell}+|a_m|^2-\lambda_m)\right\}. \qquad (6.1.14)
\end{aligned}$$

Proof All that is required is to add and multiply $\overline{C_+}$ and C_- together correctly. □

We can now calculate the neutral stability curves. Recall that these are curves along which the mth rotating wave is neutrally stable to linear perturbations of wave number ℓ; that is the eigenvalues of the matrix in (6.1.9) have zero real parts.

Theorem 6.1.1 *Given the rotating wave U_m for the semi-discrete problem* (3.6.9), *it is neutrally stable to linear perturbations of the ℓth wave, $\ell \neq 0$, along the curve described by:*

$$(S_{m+\ell} - S_{m-\ell})^2 \left\{|a_m|^4(\nu-\mu)^2 - |a_m|^2(\nu-\mu)\nu(S^2_{m+\ell} + S^2_{m-\ell} + 2(|a_m|^2 - \lambda_m))\right\} =$$
$$\left\{(1+\nu)^2(S^2_{m+\ell}S^2_{m-\ell} - S^2_{m+\ell}\lambda_m - \lambda_m S^2_{m-\ell} + \lambda_m^2) + (1+\nu\mu)(S^2_{m+\ell} + S^2_{m-\ell} - 2\lambda_m)|a_m|^2\right\}$$
$$\times \left\{S^2_{m+\ell} + S^2_{m-\ell} + 2(|a_m|^2 - \lambda_m)\right\}^2 \qquad (6.1.15)$$

where $S_{m+\ell}$ and $S_{m-\ell}$ are defined in (6.0.2) *and* (6.0.3).

Proof

The eigenvalues of the matrix in (6.1.9) are given by the roots of

$$\lambda^2 + \lambda(\overline{C_+} + C_-) + (\overline{C_+}C_-) - (1+\mu^2)|a_m|^4 = 0. \qquad (6.1.16)$$

To find the neutral stability curves we set $\lambda = 0 + i\lambda'$ and compare the real and imaginary parts

$$-\lambda'^2 + \lambda'\mathrm{Im}(\overline{C_+} + C_-) + \mathrm{Re}(\overline{C_+}C_-) - (1+\mu^2)|a_m|^4 = 0, \qquad (6.1.17)$$

and

$$\lambda'\mathrm{Re}(\overline{C_+} + C_-) + \mathrm{Im}(\overline{C_+}C_-) = 0. \qquad (6.1.18)$$

From equation (6.1.18) we may easily find λ' :

$$\lambda' = -\frac{\mathrm{Im}(\overline{C_+}C_-)}{\mathrm{Re}(\overline{C_+} + C_-)}.$$

The neutral stability curves are then found by substituting λ' back into (6.1.17) :

$$\left[\frac{\mathrm{Im}(\overline{C_+}C_-)}{\mathrm{Re}(\overline{C_+} + C_-)}\right]^2 - \frac{\mathrm{Im}\overline{C_+}C_-}{\mathrm{Re}(\overline{C_+} + C_-)}(\overline{C_+} + C_-) + \mathrm{Re}(\overline{C_+}C_-) - (1+i\mu^2)|a_m|^4. \quad (6.1.19)$$

To find (6.1.15) from equation (6.1.19) we must use the expressions for $\overline{C_+} + C_-$ and $\overline{C_+}C_-$ given in Lemma (6.1.5). After a certain amount of algebra we see that

$$\lambda' = \left\{(\nu S^2_{m+\ell} + \mu|a_m|^2 - \nu\lambda_m)(S^2_{m-\ell} + |a_m|^2 - \lambda_m)\right.$$
$$\left.-(S^2_{m+\ell} + |a_m|^2 - \lambda_m)(\nu S^2_{m-\ell} + \mu|a_m|^2 - \nu\lambda_m)\right\} \times \left\{S^2_{m+\ell} + S^2_{m-\ell} + 2(|a_m|^2 - \lambda_m)\right\}^{-1}$$
$$(6.1.20)$$

and from (6.1.17) we find the relation (6.1.15). □

Lemma 6.1.6 *Given the mth rotating wave U_m for the semi-discrete problem, it is not linearly unstable to perturbations in the* 0*th mode* ($\ell = 0$).

Proof When the perturbation is in the $\ell = 0$ mode then we notice that

$$C_+ = C_- = (1 + i\mu)|a_m|^2$$

and so we may calculate the eigenvalues of the matrix in equation (6.1.9) exactly. From (6.1.16) we see they are the roots of

$$\lambda^2 + 2\lambda|a_m|^2 = 0.$$

Hence we have the eigenvalues $\lambda = 0$ and $\lambda = -2|a_m|^2$. The result follows.□

Remarks

1. First we note that the relation for the neutral stability curves (6.1.15) converges with order $O(\Delta x^2)$ as Δx tends to zero to the relation for the continuous neutral stability curves given in equation (3.5.80). We also note that the stability result of Lemma 6.1.6 for perturbations in the zero mode is exactly the same as for the continuous case found by Doering et al [41] and quoted in Section 3.5.4.

2. The continuation code **Pitcon** [107] can be used to plot out the neutral stability curves given by equation (6.1.15). In Figures 6.3 we see the result of those computations with μ and Δx fixed at $-\sqrt{3}$ and $1/64$ respectively and ν taking the values (a) $\nu = -\sqrt{3}$, (b) $\nu = 1/\sqrt{3}$, (c) $\nu = \sqrt{3}$ and (d) $\nu = 3\sqrt{3}$. The mth wave comes into existence at the bifurcation point (m, m) and remains in existence along the horizontal from that point. The neutral stability curve for perturbations of wave number ℓ originate form the ℓth point along the diagonal. Recall that to find the stability of the mth rotating wave against linear perturbations of the ℓth we simply locate where the mth wave is on the figure and check to see if it lies above or below the neutral stability curve for the ℓth rotating wave. If it lies above then it is unstable to perturbations of the ℓth wave, and if it lies below it is stable. For example in Figures 6.3 (a) and (b) the spatially homogeneous wave ($m = 0$) is stable to all perturbations.

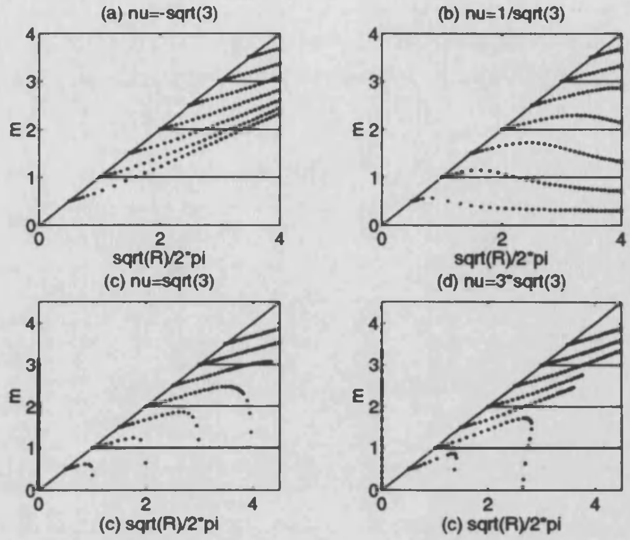

Figure 6.3: Neutral stability curves found by **Pitcon** for the parameters $\mu = -\sqrt{3}$, $\Delta x = 1/64$, (a) $\nu = -\sqrt{3}$, (b) $\nu = 1/\sqrt{3}$, (c) $\nu = \sqrt{3}$, (d) $\nu = 3\sqrt{3}$.

3. In Figure 6.4 we have plotted in the computational co-ordinates, (ie in the (k_m^2, R) plane) the neutral stability curves for $\mu = -\sqrt(3)$, $\Delta x = 1/64$ for (a) $\nu = \sqrt{3}$ and (b) $\nu = 3\sqrt{3}$. We note that **Pitcon** [107] has followed the neutral stability curves below $k_m^2 = 0$ which corresponds to m being either positive or negative. **Pitcon** [107] has also followed the curves across the $R = 0$ axis to non-physical solutions of (6.1.15). This corresponds to (6.1.15) being a quartic in R.

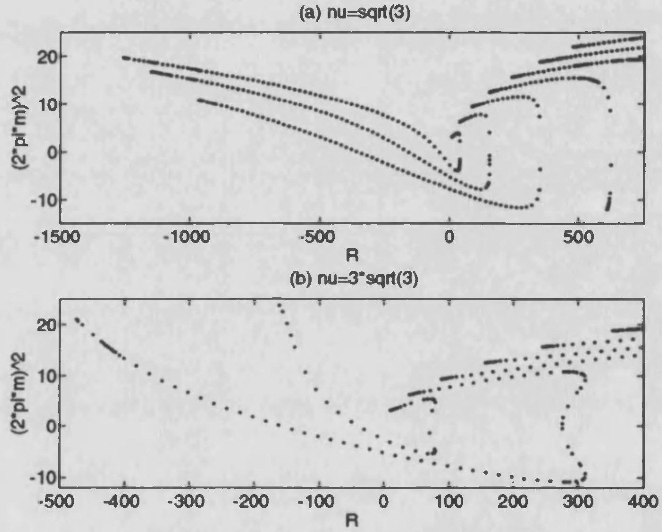

Figure 6.4: Neutral stability curves in computational co–ordinates for $\mu = -\sqrt{3}$, $\Delta x = 1/64$,(a) $\nu = \sqrt{3}$, (b) $\nu = 3\sqrt{3}$.

6.2 Discrete Rotating Waves

In this section we consider in detail rotating wave solutions for three numerical schemes. We commence with the scheme **DE** (3.6.15) with the non–linear term fully explicit and then consider the schemes **DEI** and **DI** simultaneously.

6.2.1 Rotating Waves for the scheme DE

As in the semi–discrete case we first consider existence and then examine stability.

Lemma 6.2.1 *The discrete problem* **DE** (3.6.15) *admits rotating wave solutions of the form*

$$U_m^{n+1} = \left(U_{m,0}^{n+1}, \cdots, U_{m,j}^{n+1}, \cdots, U_{m,j}^{n+1}\right)^T;$$

where

$$U_{m,j}^{n+1} = a_m e^{i(k_m j\Delta x - \omega_m (n+1)\Delta t)} \tag{6.2.21}$$

provided the amplitude a_m and period ω_m satisfy

$$\{1 - R\Delta t + \Delta t(1 + i\nu)\lambda_m\} e^{-i\omega_m \Delta t} = 1 - \Delta t(1 + i\mu)|a_m|^2. \tag{6.2.22}$$

Proof Recall the scheme **DE** written in component form:

$$\frac{U^{n+1} - U^n}{\Delta t} = RU_j^{n+1} - (1 + i\nu)\delta^2 U_j^{n+1} - (1 + i\mu)|U_j^n|^2 U_j^n. \tag{6.2.23}$$

If we substitute the jth rotating wave component (6.2.21) into (6.2.23) and compare coefficients we find

$$e^{-i\omega_m \Delta t} - 1 = R\Delta t e^{-i\omega_m \Delta t} - \Delta t(1 + i\nu)\lambda_m e^{-i\omega_m \Delta t} - \Delta t(1 + i\mu)|a_m|^2.$$

From which (6.2.22) is immediate.□

Note

By comparing real and imaginary parts relation (6.2.22) may be re–written as the coupled equations :

$$\cos(\omega_m \Delta t)\{1 - R\Delta t + \Delta t\lambda_m\} + \nu\Delta t\lambda_m \sin(\omega_m \Delta t) = 1 - \Delta t|a_m|^2 \tag{6.2.24}$$

and

$$\nu\Delta t\lambda_m \cos(\omega_m \Delta t) - \{1 - R\Delta t + \Delta t\lambda_m\} \sin(\omega_m \Delta t) = -\Delta t\mu|a_m|^2. \tag{6.2.25}$$

Lemma 6.2.2 *The discrete amplitude a_m and period ω_m defined by equations* (6.2.24) *and* (6.2.25) *converge with order $O(\Delta t + \Delta x^2)$ as $\Delta t, \Delta x \to 0$ to the relations given by equations* (3.5.77) *and* (3.5.78) *for the continuous amplitude and period.*

Proof

Expand $\cos(\omega_m \Delta t)$, $\sin(\omega_m \Delta t)$ and λ_m using Taylor's series to find the result. □

We now consider when the rotating waves come into existence.

Lemma 6.2.3 *For all $\Delta t, \Delta x > 0$ and each $m \in \mathbb{Z}$ at*

$$R_1 = \frac{1 + \Delta t \lambda_m - \sqrt{1 - \Delta t^2 \nu^2 \lambda_m^4}}{\Delta t}, \tag{6.2.26}$$

a rotating wave U_m satisfying (6.2.22) *comes into existence, and at*

$$R_2 = \frac{1 + \Delta t \lambda_m + \sqrt{1 - \Delta t^2 \nu^2 \lambda_m^4}}{\Delta t} \tag{6.2.27}$$

a rotating wave V_m also satisfying (6.2.22) *comes into existence.*

The point R_2 corresponds to a spurious starting point of spurious rotating wave solutions V_m, whereas the point R_1 converges with order $O(\Delta t + \Delta x^2)$ to the continuous starting point given in Lemma (3.5.8).

Proof The rotating waves come into existence when $|a_m|^2 = 0$. Setting $|a_m|^2 = 0$ in equation (6.2.22) we get

$$\begin{aligned} (1 - 2R\Delta t + \Delta t(1 + i\nu)\lambda_m)\, e^{-i\omega_m \Delta t} &= 1 \\ \Longrightarrow (1 - 2R\Delta t + \Delta t(1 + i\nu)\lambda_m) &= e^{i\omega_m \Delta t}. \end{aligned}$$

Equating real and imaginary parts we find

$$(1 - R\Delta t + \Delta t \lambda_m) = \cos(\omega_m \Delta t)$$

and

$$\nu \Delta t \lambda_m = \sin(\omega_m \Delta t).$$

Noting that $\sin^2(\omega_m \Delta t) + \cos^2(\omega_m \Delta t) = 1$ we obtain a quadratic relation for R

$$(1 - R\Delta t + \Delta t \lambda_m)^2 + \nu^2 \Delta t^2 \lambda_m^2 = 1, \tag{6.2.28}$$

from which we find the values of R at which rotating waves come into existence are given by R_1 (6.2.26) and R_2 (6.2.27).

We note that by Taylor's theorem

$$R_1 = \lambda_m + O(\Delta t + \Delta x^2)$$

and hence converges to the points where the continuous waves come into existence, whereas

$$R_2 \to \infty \quad \text{as } \Delta x, \Delta t \to 0$$

and are hence termed "spurious".

We now prove that the rotating waves originate from Hopf-like bifurcations of the trivial solution.

Lemma 6.2.4 *The Hopf-like bifurcation points for the trivial solution $U \equiv 0$ for the discrete problem* **DE** (3.6.15), *are given by R_1 and R_2 defined by* (6.2.26) *and* (6.2.27) *respectively.*

Proof Linearize (3.6.15) about a solution U^n to find the linear evolution equation for $\epsilon^n \in \mathbb{C}$:

$$\frac{\epsilon^{n+1} - \epsilon^n}{\Delta t} = (RI - (1+i\nu)M^{-1}A)\,\epsilon^{n+1} - (1+i\mu)\left\{2G(|U^n|^2)\epsilon^n + G((U^n)^2)\overline{\epsilon^n}\right\}, \tag{6.2.29}$$

where we recall that for $V \in \mathbb{C}^J_{\text{per}}$, $G(V)$ is defined to be the diagonal matrix with entries V_j, and $G(|V|^2)$ is taken as the diagonal matrix with entries $|V_j|^2$. Thus linearizing about the trivial solution $U \equiv 0$ we find :

$$\left\{(1 - R\Delta t)I + \Delta t(1+i\nu)M^{-1}A\right\}\epsilon^{n+1} = \epsilon^n,$$

and so

$$\epsilon^{n+1} = \left\{(1 - R\Delta t)I + \Delta t(1+i\nu)M^{-1}A\right\}^{-1}\epsilon^n.$$

The eigenvalues η_m of the operator $\left\{(1 - R\Delta t)I + \Delta t(1+i\nu)M^{-1}A\right\}^{-1}$ are given by

$$\eta_m = 1 - R\Delta t + (1+i\nu)\lambda_m.$$

For the bifurcation we require that

$$\eta_m\overline{\eta_m} = (1 - R\Delta t + \lambda_m)^2 + \nu^2\lambda_m^2 = 1,$$

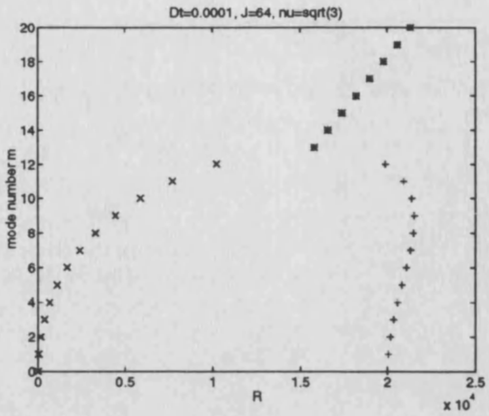

Figure 6.5: Bifurcation points of $U \equiv 0$ for the scheme **DE**: $\Delta x = 1/64$, $\nu = \sqrt{3}$, $\Delta t = 0.0001$.

which is precisely equation (6.2.28). Hence the Lemma is proved.□

In Figure 6.5 we have plotted in the (R, m) plane the bifurcation points at which the mth rotating wave comes into existence. The parameter values are $\mu = -\sqrt{3}$, $\nu = \sqrt{3}$, $\Delta x = 1/64$ and $\Delta t = 0.0001$. The points plotted by $+$ represent the spurious points R_2 and the points $\times$ are the points R_1 which converge to the true bifurcation points. Where the branch $+$ and the branch $\times$ intersect is where R takes complex values. In computations this would not be observed as R would be taken to be real. The dependence on Δt can be seen by comparing Figure 6.5 to Figure 6.6, which shows the two branches of R_1 and R_2 for the same values of ν, μ Δx but for $\Delta t = 0.00025$.

Figure 6.7 shows a plot of the rotating waves U_m, $m = 0, 1, 2, 3, 4$ for the problem **DE** in the $(R, |a_m|^2)$ plane. This was produced using the continuation code **Pitcon** [107] to solve the relationships (6.2.24) and (6.2.25), taking as an initial value the correct approximation to the bifurcation point R_1 from Lemma 6.2.3. This figure should be compared to the rotating wave bifurcation diagram for the continuous problem (Figure 3.1). Even though we have started at the bifurcation points R_1 we see the existence of spurious behaviour as the branches have a turning point and bend back on themselves. For a fixed value of $R > 0$ two rotating waves exist in the same mode with different amplitudes on the same branch. We show below that the turning point for the $m = 0$

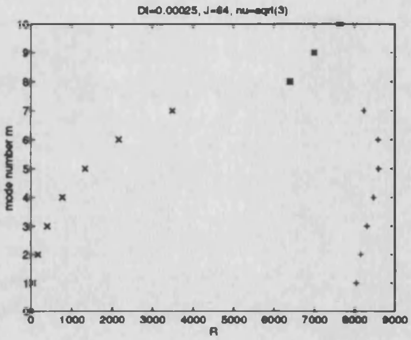

Figure 6.6: Bifurcation points of $U \equiv 0$ for the scheme **DE**: $\Delta x = 1/64$, $\nu = \sqrt{3}$, $\Delta t = 0.00025$.

wave tends to infinity as $\Delta t \to 0$, and numerical evidence indicates this is true for all the wave numbers. Furthermore we note presence of non–physical rotating wave solutions for $R < 0$. In Figure 6.8 we have plotted the spurious rotating waves V_m $m = 0, 1, 2, 3, 4$ starting at the spurious bifurcation points given by R_2.

In order to gain some further insight into Figures (6.5-6.8) let us investigate the case $m = 0$ in more detail. Figure 6.9 shows the wave $m = 0$ starting from the correct bifurcation point for $\Delta t = 10^{-3}$, $\Delta x = 1/64$, $\nu = -\sqrt{3}$ and $\mu = -\sqrt{3}$ (we have plotted both the amplitude and period against R). Figure 6.10 shows the same plot but for $\Delta t = 10^{-4}$. Note that the turning point moves as a function of Δt.

The Spatially Homogeneous Wave, ($m = 0$).

The $m = 0$ wave U_0 is given by

$$U_0^{n+1} = a_0 \left(e^{-i\omega_0(n+1)\Delta t}, \cdots, e^{-i\omega_0(n+1)\Delta t}\right)^T = a_0 e^{-i\omega_0(n+1)\Delta t} (1, \cdots, 1)^T$$

and is the spatially homogeneous wave. From equations (6.2.24) and (6.2.25) the amplitude a_0 and period ω_0 satisfy

$$\cos(\omega_0 \Delta t) \{1 - R\Delta t\} + \nu \Delta t = 1 - \Delta t |a_0|^2 \tag{6.2.30}$$

and

$$\{1 - R\Delta t\} \sin(\omega_0 \Delta t) = \Delta t \mu |a_0|^2. \tag{6.2.31}$$

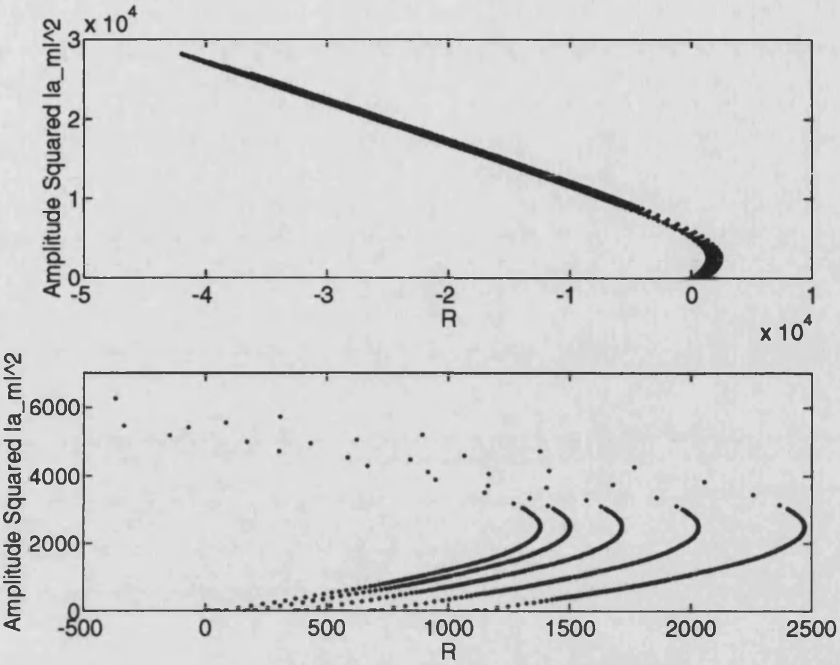

Figure 6.7: First 5 rotating waves starting from the points R_1.

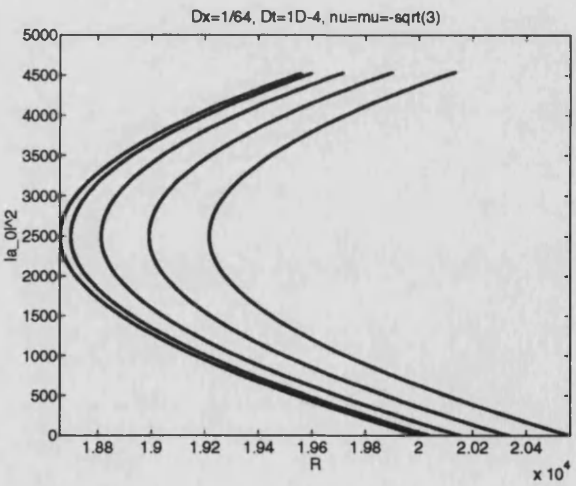

Figure 6.8: First 5 spurious rotating waves from the points R_2.

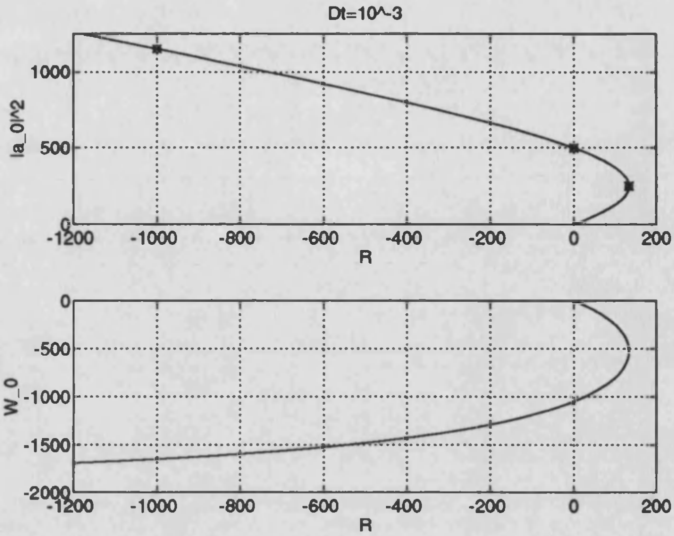

Figure 6.9: Spatially homogeneous rotating waves for **DE**, $|a_0|^2$ and ω_0 vs R ($\Delta t = 10^{-3}$).

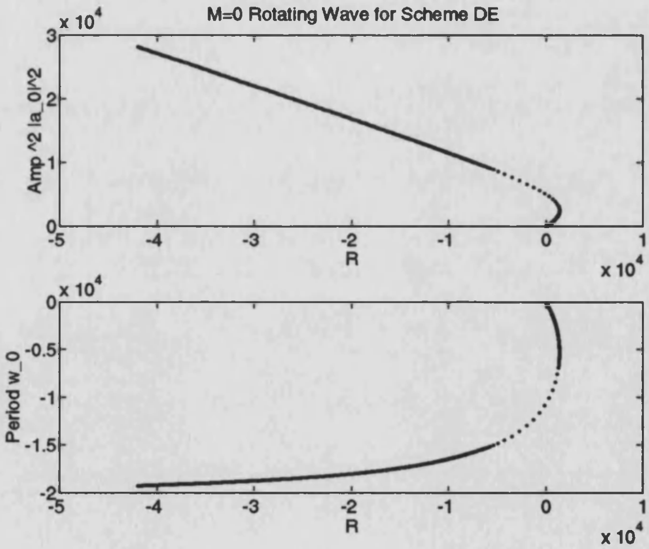

Figure 6.10: Spatially homogenous waves for **DE**, $|a_0|^2$ vs R and ω_0 vs R ($\Delta t = 10^{-4}$).

We start by seeking values of R at which these waves come into existence, that is when $|a_m|^2 = 0$. With $a_m = 0$ equations (6.2.30) (6.2.31) reduce to

$$\cos(\omega_0 \Delta t)\,(1 - R\Delta t) = 1 \tag{6.2.32}$$

and

$$\sin(\omega_0 \Delta t)\,(1 - R\Delta t) = 0, \tag{6.2.33}$$

thus we find the period ω_0 is given by

$$\omega_0 = p\pi/\Delta t$$

where $p \in \mathbb{Z}$, and that

$$(-1)^p(1 - R\Delta t) = 1.$$

Therefore the spatially homogeneous wave U_0 comes into existence at

$$R = 0 \text{ with period } \omega_0 = p\pi/\Delta t \text{ when } p \text{ is even,}$$

and the spurious spatially homogeneous wave V_m comes into existence at

$$R = \frac{2}{\Delta t} \text{ with period } \omega_0 = p\pi/\Delta t \text{ when } p \text{ is odd.}$$

These are exactly the values of R_1 and R_2 of Lemma (6.2.3) with $m = 0$.

We now seek an explicit relation between $|a_0|^2$ and R. To do this we use (6.2.32) and (6.2.33) and the trigonometric identity $\sin^2(\omega_0\Delta t) + \cos^2(\omega_0\Delta t) = 1$. This yields that

$$\mu^2\Delta t^2|a_0|^4 + (1 - \Delta t|a_0|^2)^2 = (1 - R\Delta t)^2$$

from which

$$(1 + \mu^2)\Delta t|a_0|^4 - 2|a_0|^2 + 2R - R^2\Delta t = 0.$$

Solving for $|a_0|^2$ we find

$$|a_0|^2 = \frac{1 \pm \sqrt{1 - (1 + \mu^2)\Delta t(2R - R^2\Delta t)}}{(1 + \mu^2)\Delta t}. \tag{6.2.34}$$

We now use equation (6.2.34) to check Figures 6.6 – 6.9.

- When $R = 0$ we see that either

$$|a_0|^2 = 0 \text{ or } |a_0|^2 = \frac{2}{(1+\mu^2)\Delta t}.$$

Thus for $\mu = -\sqrt{3}$ either $|a_0|^2 = 0$ or $|a_0|^2 = \frac{1}{2}\Delta t^{-1}$. These are exactly the two values at $R = 0$ we see in Figures 6.10 and in 6.9.

- When $|a_0|^2$ is a single valued function of R the descriminent of equation (6.2.34) equals zero, i.e.

$$1 - (1+\mu^2)2R\Delta t + \Delta t^2 R^2(1+\mu^2) = 0. \tag{6.2.35}$$

In Figures (6.7) – (6.9) this corresponds to the turning point for the amplitude. Solving (6.2.35) for $R\Delta t$ we get that

$$R\Delta t = 1 \pm \mu(1+\mu^2)^{-1/2}. \tag{6.2.36}$$

For example when $\mu = -\sqrt{3}$, the turning point is found at

$$R = \left(1 \pm \frac{1}{2}\sqrt{3}\right)/\Delta t.$$

The numerical value for the turning point in Figures 6.9 and 6.10 correspond to $R = \left(1 - \frac{1}{2}\sqrt{3}\right)/\Delta t$, whereas the turning point in Figure 6.8 corresponds to $R = \left(1 + \frac{1}{2}\sqrt{3}\right)/\Delta t$. We note that as $\Delta t \to 0$ both the turning points tend to infinity.

- Finally for $R < 0$ there exists amplitude a_0 and period ω_0 such that there exists a rotating wave U_m.

Example: For $R = -1/\Delta t$ and $\mu = -\sqrt{3}$ we have that

$$|a_0|^2 = \frac{1 \pm \sqrt{13}}{4\Delta t}$$

which is the value observed in the figures. The relevant period may be found thus from equation (6.2.22).

Having shown that rotating waves exist for the discrete problem **DE** we turn attention to their stability. As in section 6.1 we consider the stability of the mth wave to linear perturbations of wave number ℓ.

Lemma 6.2.5 *Let U_m^{n+1} be the rotating wave for (3.6.15) at the $n+1$ time level. Let ϵ_ℓ^{n+1}, $\epsilon_\ell^{n+1} = \left(\epsilon_{\ell,0}^{n+1}, \cdots, \epsilon_{\ell,J-1}^{n+1}\right)^T$, be an arbitrary perturbation to U_m^{n+1}. Then the linearized evolution of the jth component $\epsilon_{\ell,j}^{n+1}$ of the perturbation ϵ_ℓ^{n+1} is given by*

$$\begin{aligned}
&\left\{\epsilon_j^n - \Delta t(1+i\mu)|a_m|^2(\overline{\epsilon_j^n} + 2\epsilon_j^n)\right\}\{1 - R\Delta t + \Delta t(1+i\nu)\lambda_m\} \\
&= \left\{\epsilon_j^{n+1}(1-R\Delta t) - \frac{\Delta t}{\Delta x^2}(1+i\nu)\left(e^{-ik_m\Delta x}\epsilon_{j-1}^{n+1} - 2\epsilon_j^{n+1} + \epsilon_{j+1}^{n+1}e^{ik_m\Delta x}\right)\right\} \\
&\quad \times \{1 - \Delta t(1+i\mu)|a_m|^2\}. \qquad (6.2.37)
\end{aligned}$$

Proof For the purposes of this proof we shall omit the subscript ℓ on the perturbation. Substitute $U_{m,j}^{n+1}(1+\epsilon_j^{n+1})$ into (6.2.23) to get

$$\begin{aligned}
&(1+\epsilon_j^{n+1})e^{-i\omega_m\Delta t} - (1+\epsilon_j^n) \\
&= R\Delta t(1+\epsilon_j^{n+1})e^{-i\omega_m\Delta t} + (1+i\nu)e^{-i\omega_m\Delta t}\frac{\Delta t}{\Delta x^2}\left(e^{-ik_m\Delta x}\epsilon_{j-1}^{n+1} - 2\epsilon_j^{n+1} + \epsilon_{j+1}^{n+1}e^{ik_m\Delta x}\right) \\
&\quad - \Delta t(1+i\nu)e^{-i\omega_m\Delta t}\lambda_m - \Delta t(1+i\mu)(1+\epsilon_j^n)|a_m|^2|1+\epsilon_j^n|^2.
\end{aligned}$$

Collecting terms we see that

$$\begin{aligned}
&\left\{1+\epsilon_j^n - \Delta t(1+i\mu)(1+\epsilon_j^n)|a_m|^2|1+\epsilon_j^n|^2\right\}e^{i\omega_m\Delta t} \\
&\qquad = (1+\epsilon_j^{n+1})(1-R\Delta t) + (1+i\nu)\lambda_m\Delta t \\
&\qquad\quad - \frac{\Delta t}{\Delta x^2}(1+i\nu)\left(e^{-ik_m\Delta x}\epsilon_{j-1}^{n+1} - 2\epsilon_j^{n+1} + \epsilon_{j+1}^{n+1}e^{ik_m\Delta x}\right),
\end{aligned}$$

and we now substitute in for $e^{i\omega_m\Delta t}$ from equation (6.2.22) to get

$$\begin{aligned}
&\left\{1+\epsilon_j^n - \Delta t(1+i\mu)(1+\epsilon_j^n)|a_m|^2|1+\epsilon_j^n|^2\right\}\{1 - R\Delta t + \Delta t(1+i\nu)\lambda_m\} \\
&\times\{1 - \Delta t(1+i\mu)|a_m|^2\}^{-1} \\
&\qquad = (1+\epsilon_j^{n+1})(1-R\Delta t) + (1+i\nu)\lambda_m\Delta t \\
&\qquad\quad - \frac{\Delta t}{\Delta x^2}(1+i\nu)\left(e^{-ik_m\Delta x}\epsilon_{j-1}^{n+1} - 2\epsilon_j^{n+1} + \epsilon_{j+1}^{n+1}e^{ik_m\Delta x}\right).
\end{aligned}$$

Rearranging we find

$$\begin{aligned}
&\{1 - R\Delta t + \Delta t(1+i\nu)\lambda_m\}\left\{1+\epsilon_j^n - \Delta t(1+i\mu)(1+\epsilon_j^n)|a_m|^2|1+\epsilon_j^n|^2\right\} \\
&= \left\{(1-R\Delta t) + \epsilon_j^{n+1}(1-R\Delta t) + (1+i\nu)\lambda_m\Delta t\right. \\
&\quad \left.- \frac{\Delta t}{\Delta x^2}(1+i\nu)\left(e^{-ik_m\Delta x}\epsilon_{j-1}^{n+1} - 2\epsilon_j^{n+1} + \epsilon_{j+1}^{n+1}e^{ik_m\Delta x}\right)\right\} \times \{1 - \Delta t(1+i\mu)|a_m|^2\}.
\end{aligned}$$

All that remains to find (6.2.37) is to linearize in ϵ_j^n and perform all cancellations.□

We now wish to examine the stability of the mth wave to linear perturbations of the ℓth wave number. To achieve this set the jth component of ϵ_ℓ to be given by

$$\epsilon_{\ell,j} = \alpha_\ell^{n+1}\exp(ik_\ell j\Delta x) + \alpha_{-\ell}^{n+1}\exp(-ik_\ell j\Delta x). \tag{6.2.38}$$

Lemma 6.2.6 *Consider a perturbation ϵ_ℓ^{n+1} with components $\epsilon_{\ell,j}^{n+1}$ given by* (6.1.19) *to the mth rotating wave U_m^{n+1} with $m \neq \ell$. Then the linear evolution equations for the amplitudes α_ℓ^{n+1} and $\alpha_{-\ell}^{n+1}$ satisfy*

$$\begin{pmatrix} \overline{\alpha_\ell^{n+1}} \\ \alpha_{-\ell}^{n+1} \end{pmatrix} = \begin{pmatrix} \overline{CB_+^{-1}} & \overline{DB_+^{-1}} \\ DB_-^{-1} & CB_-^{-1} \end{pmatrix} \begin{pmatrix} \overline{\alpha_\ell^n} \\ \alpha_{-\ell}^n \end{pmatrix}, \tag{6.2.39}$$

where

$$\begin{aligned} B_{+/-} &:= \left(1 - R\Delta t + \Delta t(1+i\nu)S_{m+\ell/m-\ell}^2\right)\left(1 - \Delta t|a_m|^2(1+i\mu)\right), \\ D &:= -\Delta t(1+i\mu)|a_m|^2\left(1 - R\Delta t + \Delta t(1+i\nu)\lambda_m\right), \\ C &:= \left(1 - 2\Delta t(1+i\mu)|a_m|^2\right)\left(1 - R\Delta t + \Delta t(1+i\nu)\lambda_m\right). \end{aligned}$$

Proof Substitute the perturbation given by (6.2.38) into the linearized evolution equation (6.2.37) and compare coefficients. After some algebra we find

$$\begin{aligned} &\{\alpha_\ell^{n+1}(1 - R\Delta t) + \Delta t(1+i\nu)\alpha_\ell^{n+1}S_{m+\ell}^2\}\{1 - \Delta t|a_m|^2(1+i\mu)\} \\ = \ &\{\alpha_\ell^n - \Delta t(1+i\mu)|a_m|^2(\overline{\alpha_{-\ell}^n} + 2\alpha_\ell^n)\}\left(1 - R\Delta t + \Delta t(1+i\nu)\lambda_m\right) \end{aligned}$$

and

$$\begin{aligned} &\{\alpha_{-\ell}^{n+1}(1 - R\Delta t) + \Delta t(1+i\nu)\alpha_{-\ell}^{n+1}S_{m-\ell}\}\{1 - \Delta t|a_m|^2(1+i\mu)\} \\ = \ &\{\alpha_{-\ell}^n - \Delta t(1+i\mu)|a_m|^2(\overline{\alpha_\ell^n} + 2\alpha_{-\ell}^n)\}\left(1 - R\Delta t + \Delta t(1+i\nu)\lambda_m\right). \end{aligned}$$

All that remains is to write the equations in the appropriate matrix form.□

Lemma 6.2.7 *The eigenvalues of the matrix in* (6.2.39) *are given by*

$$\lambda_{+/-} = \frac{1}{2}\left(\overline{CB_+^{-1}}\right) \pm \frac{1}{2}\sqrt{(\overline{CB_+^{-1}})^2 - 4(|C|^2 - |D|^2)\overline{B_+^{-1}}B_-^{-1}}. \tag{6.2.40}$$

Proof Straightforward.□

Notes

By applying Taylor's theorem in Δx to $S_{m+\ell}, S_{m-\ell}$ and the eigenvalue λ_m and also applying Taylor's theorem in Δt we find that the matrix in equation (6.2.39) converges to the matrix in equation (6.1.9). The eigenvalues may be examined in a similar way.

We can now calculate the neutral stability curves. Recall that these are curves along which the mth rotating wave is neutrally stable to linear perturbations in the wave ℓ. Since we now have a discrete system we require that the eigenvalues given by (6.2.40) have modulus one.

The neutral stability curves for the scheme **DE** may be found using the continuation program **Pitcon** [107]. The amplitude a_m and period ω_m may be found using (6.2.24) and (6.2.25) and the eigenvalues found from equation (6.2.40). Imposing the conditions that

$$\lambda_- \overline{\lambda_-} = 1 \quad \text{and} \quad \lambda_+ \overline{\lambda_+} = 1$$

gives the neutral stability curves. Figures 6.11 shows the neutral stability curves for $\mu = -\sqrt{3}$, $\nu = 30\sqrt{3}$, $\Delta x = 1/128$ and $\Delta t = 10^{-6}$. These are were found using **Pitcon** [107] in the same manner as in the continuous and semi–discrete cases.

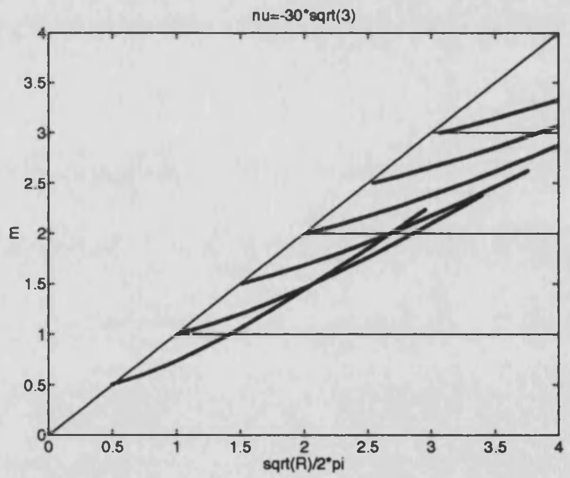

Figure 6.11: Neutral stability curves for scheme **DE** with $\nu = -30\sqrt{3}$.

However for most values of ν the "correct" curves (i.e. the ones which converge as $\Delta x, \Delta t \to 0$ to the continuous neutral stability curves) are indistinguishable. This is because **Pitcon** [107] follows the neutral stability curves of some of the spurious solutions.

6.2.2 Rotating Wave Solutions for the Schemes DI and DEI

In this section we repeat the analysis of the previous section (Section 6.2.1) for the schemes **DI** and **DEI**. Lemma (6.2.7) establishes the existence of rotating wave solutions for each scheme and that they are the same solutions for both schemes. This is of interest since solving (3.6.12) requires a non-linear solver, where as solving (3.6.16) only requires a tri–diagonal solver.

Lemma 6.2.8 *The schemes* **DI** (3.6.12) *and* **DEI** (3.6.16) *possess the same rotating wave solutions*

$$U_m^{n+1} = \left(U_{m,0}^{n+1}, \cdots, U_{m,J-1}^{n+1}\right)$$

where

$$U_{m,j}^{n+1} = a_m e^{-i(k_m j\Delta x - \omega_m (n+1)\Delta t)},$$

and the amplitude a_m and period ω_m satisfy the relation

$$\{1 - R\Delta t + \Delta t(1+i\nu)\lambda_m + \Delta t(1+i\mu)|a_m|^2\} = \exp(i\omega_m \Delta t). \qquad (6.2.41)$$

Proof Recall **DI** (3.6.12) and **DEI** (3.6.16) in component form, that is

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = RU_j^{n+1} - (1+i\nu)\delta^2 U_j^{n+1} - (1+i\mu)|U_j^{n+1}|^2 U_j^{n+1} \qquad (6.2.42)$$

and

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = RU_j^{n+1} - (1+i\nu)\delta^2 U_j^{n+1} - (1+i\mu)|U_j^{n+1}|^2 U_j^n \qquad (6.2.43)$$

respectively. Notice that if we substitute the jth component of the rotating wave into (6.2.42) and (6.2.43) then, because the amplitude remains constant, we find in both cases that

$$e^{-i\omega_m\Delta t} - 1 = R\Delta t e^{-i\omega_m\Delta t} - \Delta t(1+i\nu)\lambda_m e^{-i\omega_m\Delta t} - \Delta t(1+i\mu)e^{-i\omega_m\Delta t}|a_m|^2.$$

Equation (6.2.41) follows immediately.□

Note

By comparing real and imaginary parts, (6.2.41) may be written as the coupled equations

$$\cos(\omega_m \Delta t) = 1 - R\Delta t + \lambda_m \Delta t + \Delta t |a_m|^2 \tag{6.2.44}$$

and

$$\sin(\omega_m \Delta t) = \nu \Delta t \lambda_m + \mu \Delta t |a_m|^2. \tag{6.2.45}$$

Lemma 6.2.9 *The discrete amplitude a_m and period ω_m defined by equations* (6.2.44) *and* (6.2.45) *converge with order $O(\Delta t + \Delta x^2)$ as $\Delta t, \Delta x \to 0$ to the relations given by equations* (3.5.77) *and* (3.5.78) *for the continuous amplitude and period.*

Proof

Expand $\cos(\omega_m \Delta t)$, $\sin(\omega_m \Delta t)$ and λ_m using Taylors series to find the result. □

Lemma 6.2.10 *For all $\Delta t, \Delta x > 0$ and each $m \in \mathbb{Z}$ at R_1 given by* (6.2.26) *a rotating wave U_m satisfying* (6.2.41) *comes into existence, and at R_2 defined by* (6.2.27) *a rotating wave V_m also satisfying* (6.2.41) *comes into existence.*

The point R_2 corresponds to a spurious starting point of rotating wave solutions V_m, whereas the point R_1 converges with order $O(\Delta t + \Delta x^2)$ to the continuous starting point given in Lemma (3.5.8).

Proof The rotating waves come into existence when $|a_m|^2 = 0$. Setting $|a_m|^2 = 0$ in equation (6.2.41) we get

$$\begin{aligned} (1 - 2R\Delta t + \Delta t(1 + i\nu)\lambda_m)\, e^{-i\omega_m \Delta t} &= 1 \\ \Longrightarrow (1 - 2R\Delta t + \Delta t(1 + i\nu)\lambda_m) &= e^{i\omega_m \Delta t}. \end{aligned}$$

This is exactly the relationship we found in Lemma (6.2.3), the proof is now the same as in Lemma (6.2.3). □

Lemma 6.2.11 *The Hopf-like bifurcation points for the trivial solution $U \equiv 0$ for the discrete problem* **DI** (3.6.12) *or* **DEI** (3.6.16), *are given by R_1 and R_2 defined by* (6.2.26) *and* (6.2.27) *respectively.*

Proof Linearize (3.6.12) about a solution U^n to find the linear evolution equation for $\epsilon^n \in \mathbb{C}$:

$$\frac{\epsilon^{n+1}-\epsilon^n}{\Delta t} = (RI-(1+i\nu)M^{-1}A)\,\epsilon^{n+1} - (1+i\mu)\left\{2G(|U^{n+1}|^2)\epsilon^{n+1} + G((U^{n+1})^2)\overline{\epsilon^{n+1}}\right\}. \tag{6.2.46}$$

Linearize (3.6.16) about a solution U^n to find the linear evolution equation for $\epsilon^n \in \mathbb{C}$:

$$\begin{aligned}\frac{\epsilon^{n+1}-\epsilon^n}{\Delta t} &= (RI-(1+i\nu)M^{-1}A)\,\epsilon^{n+1} \\ &- (1+i\mu)\left\{G(U^{n+1}\overline{U^n})\epsilon^n + G(|U^n|^2)\epsilon^{n+1} + G(U^nU^{n+1})\overline{\epsilon^n}\right\},\end{aligned} \tag{6.2.47}$$

where we recall that for $V \in \mathbb{C}^J_{\text{per}}$, $G(V)$ is defined to be the diagonal matrix with entries V_j, and $G(|V|^2)$ is taken as the diagonal matrix with entries $|V_j|^2$. Thus linearizing about about the trivial solution we find in both cases

$$\{(1-R\Delta t)I + \Delta t(1+i\nu)M^{-1}A\}\,\epsilon^{n+1} = \epsilon^n,$$

and so

$$\epsilon^{n+1} = \left\{(1-R\Delta t)I + \Delta t(1+i\nu)M^{-1}A\right\}^{-1}\epsilon^n.$$

The eigenvalues η_m of the operator $\left\{(1-R\Delta t)I + \Delta t(1+i\nu)M^{-1}A\right\}^{-1}$ are given by

$$\eta_m = 1 - R\Delta t + (1+i\nu)\lambda_m.$$

For the bifurcation we require that

$$\eta_m\overline{\eta_m} = 1,$$

which is precisely equation (6.2.28). Hence the Lemma is proved.□

Remarks

1. We refer to Figures 6.5 and 6.6 for plots of the spurious points R_2 and "correct" bifurcation points R_1.

2. In Figure 6.12 we have used **Pitcon** [107] to solve (6.2.41) to find the $m = 0, 1, 2, 3, 4$ rotating waves in the $(R, |a_m|^2)$ plane.

   The parameter values taken were $\mu = -\sqrt{3}$, $\nu = -\sqrt{3}$ and $\Delta x = 1/64$, $\Delta t = 10^{-4}$. We note that for the schemes **DI** and **DEI** the mth rotating wave exists for

$R_1 \leq R \leq R_2$, i.e. between the "correct" point and the "spurious" bifurcation point. This figure should be compared with the previous bifurcation diagrams, Figures 3.1, 6.2 and 6.5. We see that although there is some spurious behaviour, provided R is not taken too large (for example for $0 < R < 10^4$) we can expect some reasonable behaviour.

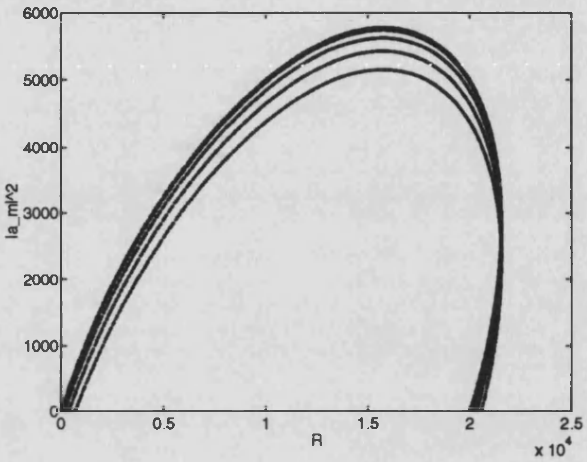

Figure 6.12: First five rotating waves for **DI** and **DEI**. $\Delta t = 10^{-4}$, $\Delta x = 1/64$, $\mu = \nu = -\sqrt{3}$.

The Spatially Homogeneous Wave, $m = 0$.

Let us examine the $m = 0$ case in closer detail. The U_0 wave is given by

$$U_0 = a_0 e^{-i\omega_0 \Delta t} (1, \cdots, 1)^T ,$$

and is spatially homogeneous. If we set $m = 0$ in equations (6.2.44) and (6.2.45) we find

$$\cos(\omega_0 \Delta t) = 1 - R\Delta t + \Delta t |a_0|^2 \text{ and } \sin(\omega_0 \Delta t) = \mu \Delta t |a_m|^2.$$

Using the trigonometric identity $\sin^2(\omega_0 \Delta t) + \cos^2(\omega_0 \Delta t) = 1$ we find

$$(1 - R\Delta t + \Delta t |a_0|^2)^2 + \left(\Delta t^2 \mu^2 |a_0|^2\right) = 1$$

and hence,

$$\Delta t^2 |a_0|^4 (1 + \mu^2) + 2(1 - R\Delta t)\Delta t |a_0|^2 - 2R\Delta t + R^2 \Delta t^2 = 0.$$

Thus solving for $|a_0|^2$ we see

$$\Delta t|a_0|^2 = \frac{R\Delta t - 1}{1+\mu^2} \pm \frac{1}{1+\mu^2}\sqrt{1-\mu^2(R^2\Delta t^2 - 2R\Delta t)}. \tag{6.2.48}$$

We can use this relation to verify points on Figures 6.12 and 6.13.

- When $R = 0$ from (6.2.48) we have that either

$$|a_0|^2 = 0 \quad \text{or } |a_0|^2 = \frac{-2}{(1+\mu^2)\Delta t}.$$

  Clearly $a_0 = 0$ corresponds to a continuous solution, whereas the other is clearly not a physical solution.

- Equation (6.2.48) can be used to determine the values of R for which $|a_0|^2$ is single valued. This is when the descriminent in (6.2.48) equals zero, i.e.

$$1 - \mu^2 R^2 \Delta t^2 + 2\mu^2 R\Delta t = 0.$$

  Solving for $R\Delta t$ we find that

$$R\Delta t = 1 \pm \sqrt{1 + 1/\mu^2}. \tag{6.2.49}$$

  For example: when $\mu = \sqrt{3}$ we find from (6.2.49) that

$$R = 1 - \frac{2}{3}\sqrt{3} \quad \text{or} \quad R = 1 - \frac{2}{3}\sqrt{3} \ < \ 0. \tag{6.2.50}$$

  These are exactly the turning points we see in Figure 6.13.

We now consider the stability of the mth rotating wave to linear perturbations of wave number ℓ.

Lemma 6.2.12 *Let U_m^{n+1} be a rotating wave as in Lemma 6.2.8 and let $\epsilon_\ell^{n+1} = \left(\epsilon_{\ell,0}^{n+1}, \cdots, \epsilon_{\ell,J-1}^{n+1}\right)^T$ be an arbitrary perturbation to U_m^{n+1}. Then the linearized evolution of the jth component $\epsilon_{\ell,j}^{n+1}$ of the perturbation ϵ_ℓ^{n+1} is given by*

$$\begin{aligned}
&\epsilon_{\ell_j}^{n}\left(1 - R\Delta t + \Delta t\lambda_m + \Delta t(1+i\mu)|a_m|^2\right) \\
&\quad = \ \epsilon_{\ell_j}^{n+1}(1 - R\Delta t) + \left(2\epsilon_{\ell_j}^{n+1} + \overline{\epsilon_{\ell_j}}^{n+1}\right)\Delta t(1+i\mu)|a_m|^2 \\
&\quad - \ \frac{\Delta t}{\Delta x^2}\left\{e^{-ik_m\Delta x}\epsilon_{\ell_{j-1}}^{n+1} - 2\epsilon_{\ell_j}^{n+1} + \epsilon_{\ell_{j+1}}^{n+1}e^{ik_m\Delta x}\right\}.
\end{aligned} \tag{6.2.51}$$

Proof For the purposes of the proof we omit the subscript ℓ. Substitute $U_{m,j}^{n+1}(1+\epsilon_j)$ into either (6.2.43) of (6.2.42) to get:

$$\begin{aligned}(1+\epsilon_j^{n+1})e^{-i\omega_m\Delta t}-(1+\epsilon_j^n) &= R\Delta t(1+\epsilon_j^{n+1})e^{-i\omega_m\Delta t}-\Delta t(1+i\nu)\lambda_m \\ &\quad -\frac{\Delta t}{\Delta x}(1+i\nu)e^{-i\omega_{\Delta t}}\left\{e^{-ik_m\Delta x}\epsilon_{j-1}^{n+1}-2\epsilon_j^{n+1}+e^{ik_m\Delta x}\epsilon_{j+1}^{n+1}\right\} \\ &\quad -\Delta t(1+i\mu)\left(1+\epsilon_j^{n+1}\right)|1+\epsilon_j^{n+1}|^2|a_m|^2e^{-i\omega_m\Delta t},\end{aligned}$$

which after re-arrangement becomes

$$\begin{aligned}(1+\epsilon_j^n)e^{i\omega_m\Delta t} &= (1+\epsilon_j^{n+1})(1-R\Delta t)+(1+\epsilon_j^{n+1})\Delta t(1+i\mu)|1+\epsilon_j^{n+1}|^2|a_m|^2 \\ &\quad +\Delta t(1+i\nu)\lambda_m-\frac{\Delta t}{\Delta x^2}(1+i\nu)\left\{e^{-ik_m\Delta x}\epsilon_{j-1}^{n+1}-2\epsilon_j^{n+1}+e^{ik_m\Delta x}\epsilon_{j+1}^{n+1}\right\}.\end{aligned}$$

Now substitute in from (6.2.41) and perform possible cancellations to get equation (6.2.51). □

We now wish to examine the stability of the mth rotating wave to linear perturbations in the ℓth mode. To achieve this we set the jth component of the perturbation ϵ_ℓ to be

$$\epsilon_{\ell_j}=\alpha_\ell\exp(ik_\ell j\Delta x)+\alpha_{-\ell}\exp(ik_\ell j\Delta x) \tag{6.2.52}$$

for $j=0,\cdots,J-1$.

Lemma 6.2.13 *Consider a perturbation ϵ_ℓ^{n+1} with components $\epsilon_{\ell,j}^{n+1}$ given by* (6.2.52) *to the mth rotating wave U_m^{n+1} with $m\neq\ell$. Then the linear evolution equations for the amplitudes α_ℓ and $\alpha_{-\ell}$ satisfy*

$$\begin{pmatrix}\overline{\alpha_\ell^{n+1}}\\ \alpha_{-\ell}^{n+1}\end{pmatrix}=\frac{|B|^{-2}}{(C_+\overline{C_-}-|D|^2)}\begin{pmatrix}\overline{C_-B} & -DB\\ -\overline{DB} & C_+B\end{pmatrix}\begin{pmatrix}\overline{\alpha_\ell^n}\\ \alpha_{-\ell}^n\end{pmatrix}, \tag{6.2.53}$$

where,

$$D:=\Delta t(1-i\mu)|a_m|^2, \tag{6.2.54}$$

$$B^{-1}:=1-R\Delta t+\Delta t(1-i\nu)\lambda_m+\Delta t(1-i\mu)|a_m|^2, \tag{6.2.55}$$

and

$$C_{+/-}:=1-R\Delta t+\Delta t(1-i\nu)S_{+/-}+2\Delta t(1-i\mu)|a_m|^2. \tag{6.2.56}$$

Proof Substitute (6.2.52) into (6.2.53) and compare coefficients to get

$$\alpha_{-\ell}^{n+1}(1-R\Delta t)+(2\alpha_{-\ell}^{n+1}+\overline{\alpha_{\ell}^{n+1}})\Delta t(1+i\mu)|a_m|^2+\Delta t(1+i\nu)\alpha_{-\ell}^{n+1}S_{m-\ell}$$
$$=\alpha_{-\ell}^{n}\left\{1-R\Delta t+\Delta t(1+i\nu)\lambda_m+\Delta t(1+i\mu)|a_m|^2\right\} \quad (6.2.57)$$

and

$$\alpha_{\ell}^{n+1}(1-R\Delta t)+(2\alpha_{\ell}^{n+1}+\overline{\alpha_{-\ell}^{n+1}})\Delta t(1+i\mu)|a_m|^2+\Delta t(1+i\nu)\alpha_{\ell}^{n+1}S_{m+\ell}$$
$$=\alpha_{\ell}^{n}\left\{1-R\Delta t+\Delta t(1+i\nu)\lambda_m+\Delta t(1+i\mu)|a_m|^2\right\}. \quad (6.2.58)$$

Thus in matrix form we get:

$$\begin{pmatrix} \overline{\alpha_{\ell}^{n}} \\ \alpha_{-\ell}^{n} \end{pmatrix} = \begin{pmatrix} C_{+}B & DB \\ \overline{DB} & \overline{C_{-}B} \end{pmatrix} \begin{pmatrix} \overline{\alpha_{\ell}^{n+1}} \\ \alpha_{-\ell}^{n+1} \end{pmatrix} \quad (6.2.59)$$

where B, C, D are given in equations (6.2.55)-(6.2.57). All that remains to recover equation (6.2.54) is to invert the matrix in (6.2.59).□

We can now calculate the the neutral stability curves by finding where the eigenvalues of the matrix in (6.2.54) have modulus one.

Lemma 6.2.14 *The eigenvalues of the matrix in* (6.2.54) *are given by :*

$$\lambda_{+/-}=\frac{1}{2}\left(\overline{CB_{+}^{-1}}\right)\stackrel{+}{-}\frac{1}{2}\sqrt{(\overline{CB_{+}^{-1}})^2-4(|C|^2-|D|^2)\overline{B_{+}^{-1}}B_{-}^{-1}}. \quad (6.2.60)$$

Proof Straightforward.□

Note

- By applying Taylor's theorem to $S_{m+\ell}, S_{m-\ell}$ and to the eigenvalue λ_m we find that the matrix in equation (6.2.54) and its eigenvalues λ_+ and λ_- given by (6.2.55) converge to the matrix in equation (6.1.9) and to its eigenvalues (6.1.16).

We can now calculate the neutral stability curves. Recall that these are curves along which the mth rotating wave is neutrally stable to linear perturbations in the wave ℓ. Since we now have a discrete system we require that the eigenvalues have modulus 1. Using the same techniques as in section (6.2.1) we are able to plot the neutral stability curves. For $\Delta t = 0.0001$ and $\Delta x = 1/128$ in Figure 6.14 we have plotted the neutral stability curves for the same parameter values as the continuous case (see Figure 3.2) and semi–discrete case (see Figure 6.3). There is clearly an excellent agreement between the three figures.

In Figure 6.15 we have plotted the neutral stability curves for $\mu = -\sqrt{3}$, (a) $\nu = \sqrt{3}$ and (b)$\nu = \sqrt{3}$, with $\Delta x = 1/128$ and $\Delta t = 10^{-5}$ in the computational coordinates. Again we see that **Pitcon** [107] follows the neutral stability curves to a non-physical regime (to the left half plane given by $R < 0$).

Finally we note that although the two schemes **DI** and **DEI** have the same rotating waves and neutral stability curves the nonlinear stability of the waves may be different for the different schemes.

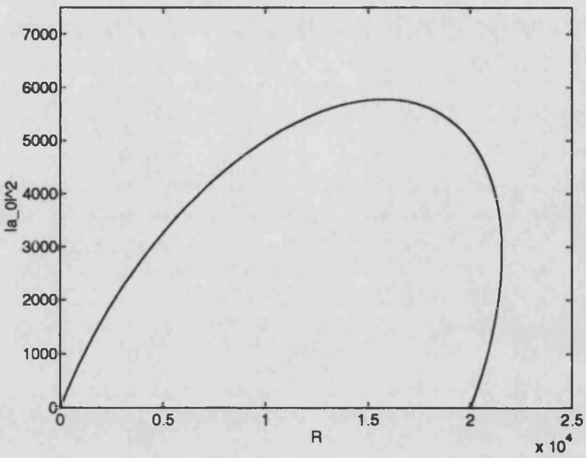

Figure 6.13: Spatially homogeneous wave for for **DI** and **DEI**. $\Delta t = 10^{-4}$, $\Delta x = 1/64$, $\mu = \nu = -\sqrt{3}$.

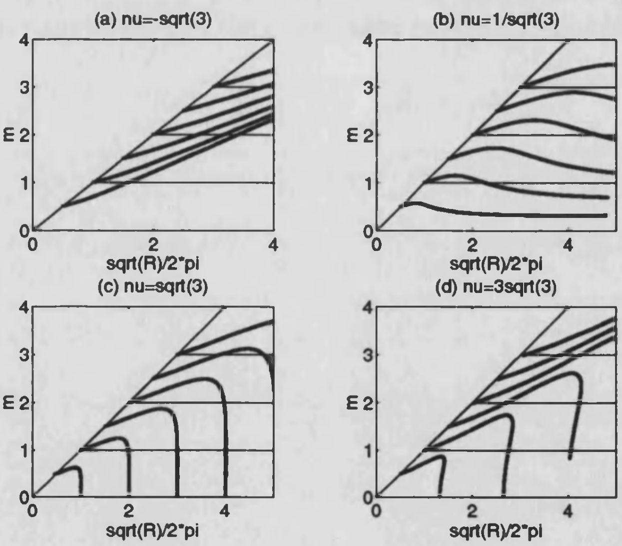

Figure 6.14: Neutral stability curves for **DI** and **DEI**. $\Delta x = 1/128$, $\Delta t = 10^{-5}$, $\mu = -\sqrt{3}$ and (a) $\nu = -\sqrt{3}$, (b) $\nu = 1/\sqrt{3}$, (c) $\nu = \sqrt{3}$ and (d) $\nu = 3\sqrt{3}$.

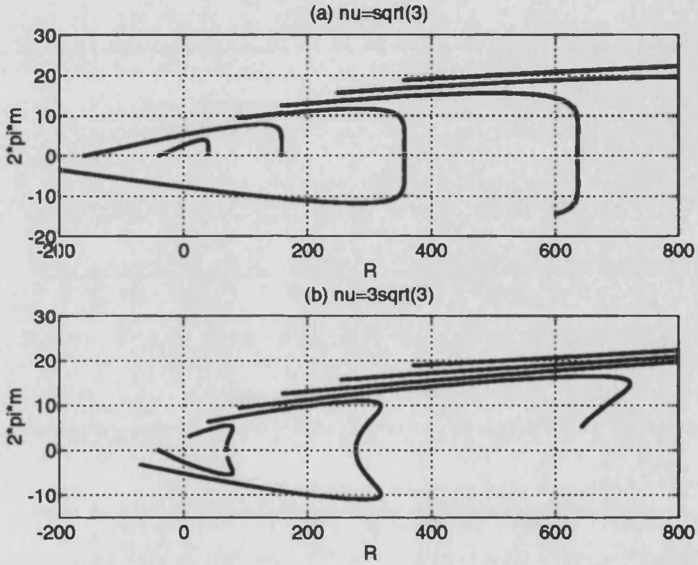

Figure 6.15: Neutral stability curves plotted in the computational co-ordinates for $\Delta x = 1/128$, $\Delta t = 10^{-5}$, $\mu = -\sqrt{3}$, (a) $\nu = \sqrt{3}$ & (b)$\nu = \sqrt{3}$.

Chapter 7

Heteroclinic Connections in the Ginzburg–Landau equation

In this chapter we turn our attention to the complex Ginzburg–Landau equation with periodic and Dirichlet boundary conditions and to its approximation by the spectral method. The eventual aim is to examine directly the structure of the global attractor. We review some relevant analytic work on the global attractor by Mischaikow and Morita [97] as well as related numerical results, principally found by Bai et al [3]. Using the numerical continuation code AUTO [37] we re–examine the bifurcation structure of the problem with both periodic and Dirichlet boundary conditions. We conclude with some preliminary results on the direct approximation of the attractor.

Notation: For this chapter we re–introduce the distinction between the spaces L^2 and L^2_{per}.

7.1 Galerkin Approximation

Consider the abstract evolution equation

$$u_t + Au = f(u) \tag{7.1.1}$$

$$u^0 = u(0)$$

in a Hilbert space X with inner–product $\langle \bullet, \bullet \rangle$ and induced norm $\| \bullet \|_X^2 = \langle \bullet, \bullet \rangle$. Suppose that A is a sectorial linear operator with eigenvalues $\{\lambda_k\}$ and orthonormal eigenvectors $\{\psi_k\}$. Write the solution u at time t as the Fourier series

$$u(t) = \sum_k a_k(t)\psi_k, \quad u(0) = \sum_k a_k(0)\psi_k, \tag{7.1.2}$$

then the Fourier coefficients a_k satisfy

$$\frac{d}{dt}a_k + \lambda_k a_k = \langle f(u), \psi_k\rangle, \quad a_k(0) = \langle u^0, \psi_k\rangle \ \forall k. \tag{7.1.3}$$

Now let $P_m : X \to X$ be the orthogonal projection onto the span$(\psi_1, \cdots, \psi_m)$ and let $Y = P_m X$. The *spectral* or *Galerkin* approximation of (7.1.1) is to find $U \in Y$ solving

$$\frac{dU}{dt} + AU = P_m f(U); \ U^0 = P_m u^0, \tag{7.1.4}$$

which yields the following system of ordinary differential equations

$$\frac{da_k}{dt} + \lambda_k a_k(t) = \langle f(U), \psi_k\rangle; \quad a_k(0) = \langle U^0, \psi_k\rangle \quad \forall k. \tag{7.1.5}$$

We note that different projections yield different spectral approximations. For further details on spectral methods we refer the reader to Gottlieb and Orszag [58] and in the particular context of the Ginzburg–Landau equation we refer the reader to Yang [139, 140].

We now state a theorem which relates the Lyapunov exponents of a system to the Lyapunov exponents for the system of Fourier coefficients.

Theorem 7.1.1 *When they exist we let $\{\mu_i^s\}$ be the Lyapunov exponents for the system 7.1.4 and let $\left\{\mu_i^f\right\}$ be the Lyapunov exponents for* (7.1.5)*. Then, after some possible re–ordering, we have*

$$\mu_i^s = \mu_i^f \qquad i = 1, \cdots, m.$$

Proof Re–write the system (7.1.4) as

$$\frac{dU}{dt} = F(U) \tag{7.1.6}$$

where $F(v) := -Av + f(v)$ for $v \in X$ and re–write the system (7.1.5) as

$$\frac{da}{dt} = G(a) \tag{7.1.7}$$

where $a = (a_1, \cdots, a_m)^T$ and

$$G(a(t)) = \begin{pmatrix} g_1(a(t)) \\ \vdots \\ g_m(a(t)) \end{pmatrix} = \begin{pmatrix} \langle F(U(t)), \psi_1\rangle \\ \vdots \\ \langle F(U(t)), \psi_m\rangle \end{pmatrix}.$$

Let the linearized equation for (7.1.6) be given by

$$\frac{dV}{dt} = DF[U(t)]V, \quad V \in X,$$

and expand V as a Fourier series with Fourier coefficients $\{b_k\}$ satisfying

$$\frac{db_k}{dt} = \langle DF[U(t)]V, \psi_k\rangle \quad \forall k = 1, \cdots, m. \tag{7.1.8}$$

Now consider the linearization evolution equation for (7.1.7) given by

$$\frac{dc}{dt} = dG[a(t)]c,$$

where $c = (c_1, \cdots, c_m)^T$. Then for all $k = 1, \cdots, m$ we have that

$$\begin{aligned} \frac{dc_k}{dt} &= \sum_j \frac{\partial g_k}{\partial a_j} c_j = \sum_j \left\langle DF[U(t)]\frac{\partial U}{\partial a_j}, \psi_k \right\rangle c_j \\ &= \sum_j \langle DF[U(t)]\psi_j, \psi_k\rangle c_j = \sum_j \langle DF[U(t)]\psi_j c_j, \psi_k\rangle \\ &= \left\langle DF[U(t)] \sum_j \psi_j c_j, \psi_k \right\rangle = \langle DF[U(t)]V, \psi_k\rangle. \end{aligned} \tag{7.1.9}$$

Noting that (7.1.8) and (7.1.9) are the same equation the result follows from the definition of the Lyapunov exponents. □

7.2 Numerical Techniques for Heteroclinic Connections

Suppose we were given two equilibrium solutions $\widetilde{U}_-$ and $\widetilde{U}_+$ to the continuous dynamical system 7.1.1 and suppose that the global stable and unstable manifolds exist for each equilibrium. Recall that for a connection to exist we require that

$$W^u(\widetilde{U}_-) \bigcap W^s(\widetilde{U}_+) \neq \phi.$$

Furthermore when $\widetilde{U}_-$ and $\widetilde{U}_+$ are hyperbolic the dimension of the manifold of heteroclinic connections is given by

$$\dim \left[W^u(\widetilde{U}_-) \bigcap W^s(\widetilde{U}_+) \right].$$

First we remark that if the stationary solution $\widetilde{U}_+$ is stable then in general the connections may be found by solving the initial value problem, as for example in Figures 4.4–4.7. In general this approach will fail when the equilibrium $\widetilde{U}_+$ is unstable.

However if the manifold of heteroclinic connections lies in a particular subspace, such as a particular Fourier subspace, then this may be exploited to find connections to unstable equilibriums by an initial value solver. We have already seen that such connections exist as the monochromatic waves lie in just one Fourier mode (see Section 3.4). This may be observed in sections 7.3.1 and 7.4.2 and was exploited in the work of Bai et al [3] to obtain starting solutions for the general technique sketched below.

Heteroclinic connections for equation (7.1.1) are given by solutions to the boundary value problem

$$\left.\begin{array}{c} u_t + Au = f(u), \\ \lim_{t\to-\infty} u(x,t) = \widetilde{U}_- \ \ \& \ \ \lim_{t\to\infty} u(x,t) = \widetilde{U}_+; \end{array}\right\} \tag{7.2.10}$$

defined on the infinite interval. We note that in order for the problem to be well posed the phase must be fixed.

Suppose (7.2.10) is now approximated by a Galerkin method to get the following system of ordinary differential equations on the infinite interval:

$$\left.\begin{array}{ccc} \frac{d\alpha}{dt} & = & G(\alpha), \\ \lim_{t\to-\infty} \alpha(x,t) = \tilde{\alpha}_- & \& & \lim_{t\to\infty} \alpha(x,t) = \tilde{\alpha}_+. \end{array}\right\} \tag{7.2.11}$$

This problem is then truncated to the finite interval $[T_-, T_+]$ to get

$$\frac{d\alpha}{dt} = G(\alpha), \quad t \in [T_-, T_+] \tag{7.2.12}$$

$$L_{+u}(\alpha(T_+) - \tilde{\alpha}_+) = 0 \tag{7.2.13}$$

$$L_{-u}(\alpha(T_-) - \tilde{\alpha}_-) = 0 \tag{7.2.14}$$

$$\Psi(\alpha) = 0, \tag{7.2.15}$$

where the operator L_{+u} projects onto the unstable manifold of $\tilde{\alpha}_+$ and L_- projects onto the stable manifold of $\tilde{\alpha}_-$. These boundary conditions are known as projection boundary conditions and were proposed by Beyn [14] for computing connections for ordinary differential equations. Similar methods are discussed by Doedel and Friedman see for example [38] and [99]. Equation 7.2.15 contains a phase fixing condition and any determining conditions. Determining conditions are required when the dimension of the manifold of heteroclinic connections is greater than 1 to ensure the problem is well posed and are used to parametrize the family of connections. For further details on the theory and implementation see, for example, [13, 14, 37, 38, 61, 62, 99, 85, 91, 90].

Recently these techniques have been extended by [4] to the case when $\widetilde{U}_+$ is an unstable periodic solution. This is achieved by considering the connections for a map arising from the continuous problem which has the periodic orbit as a fixed point.

7.3 Ginzburg–Landau with Periodic Bounday Conditions

For the Ginzburg-Landau equation (3.1.1) with periodic boundary conditions we have the following spectral approximation. For fixed $M \in \mathbb{N}, M < \infty$ let $U(t)$ be the Galerkin approximation

$$U(t) = \sum_{m=-M}^{M} a_m(t)\Psi_m.$$

The Fourier modes satisfy, after a certain amount of algebra :

$$\frac{d}{dt}a_m = Ra_m - (1+i\nu)\Lambda_m a_m(t) - (1+i\mu) \sum_{\substack{j+k=\ell+m \\ j,k,\ell=-M}}^{M} a_j a_k \overline{a_\ell}, \quad m = -M, \cdots, M. \tag{7.3.16}$$

We note that the spectral method is a natural way to discretize the Ginzburg–Landau equation with periodic boundary conditions. In section 3.4 we showed the existence of an inertial manifold for the continuous equation and hence that high wave numbers were slaved to low wave numbers. It would be possible to make explicit use of this and consider a non–linear Galerkin method however we do not pursue that here for the following reason. In section 3.4 it was also established that the solutions lie in a Gevrey class of regularity $\tau > 0$. The existence of an inertial manifold and the Gevrey regularity result both indicate that the spectral method yields a good approximation. In particular the Gevrey result shows that high wave numbers are damped exponentially - and hence we may consider neglecting them all together. Indeed it is proposed in [82] that for the case when the solutions are in a Gevrey class there is no advantage to using a non–linear Galerkin method over a standard Galerkin method.

In addition to the exponential damping of high modes we expect from the heuristic argument below that only a few modes are required to capture much of the dynamics.

We have discussed at length the rotating wave solutions (see sections 3.5.4, 4.5 and Chapter 6.2), each of which corresponds to a solution in a particular Fourier mode. The connections to that rotating wave are also contained in that Fourier mode and are given by the monochromatic waves (or Stokes solutions). Thus for a fixed value of R the minimum number of modes to take for a reasonably accurate solution may be easily determined. If at R there exist m rotating waves then we should take at least m Fourier modes in our spectral approximation. This gives not only all the rotating waves present but also the connections between them.

We recall from section 3.5.4 that the monochromatic waves (3.5.81-3.5.82) are exact heteroclinic connections for the Ginzburg–Landau equations and that the connection to the mth rotating wave is in the mth Fourier mode.

7.3.1 Numerical Results

We present in this section a selection of numerical results for the complex Ginzburg–Landau equation approximated by the Galerkin method. In each computation the number of Fourier modes were chosen according to the heuristic argument above with typically two extra modes. We note that the same qualitative dynamics was observed for far more accurate solutions. The standard fourth order Runge–Kutta scheme was used to integrate the system of ordinary differential equations given by (7.3.16). We calculated the associated largest Lyapunov exponent using the method described in section 2.1.1.

We do not present the results of computations of heteroclinic connections from the trivial solution to stable periodic orbits since these are easily computed. We simply note that similar results were found to those presented in section 4.5 for the finite difference schemes.

Using the spectral method we are able to compute examples of heteroclinic connections between unstable solutions and hence find numerically the monochromatic wave solutions. A selection of results for these computations are presented in Figures 7.1–7.4 for various values of R, ν and μ. These were found by perturbing the trivial solution in the Fourier subspace associated with the connection.

For $R = 50$, $\nu = -\sqrt{3}$ and $\mu = \sqrt{3}$ we have computed the connection from the trivial solution to the first rotating wave which is unstable. In Figure 7.1 we have plotted the Fourier modes against time. The associated largest Lyapunov exponent is positive and is shown in Figure 7.2 in which we have also plotted the solution reconstructed from the Fourier modes.

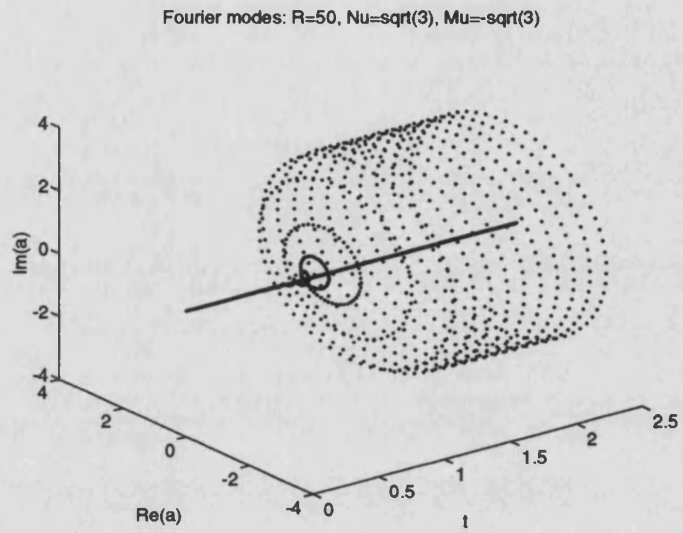

Figure 7.1: Connection in Fourier space: $U \equiv 0$ to unstable spatially homogeneous rotating wave. $R = 50$, $\nu = -\sqrt{3}$, $\mu = \sqrt{3}$.

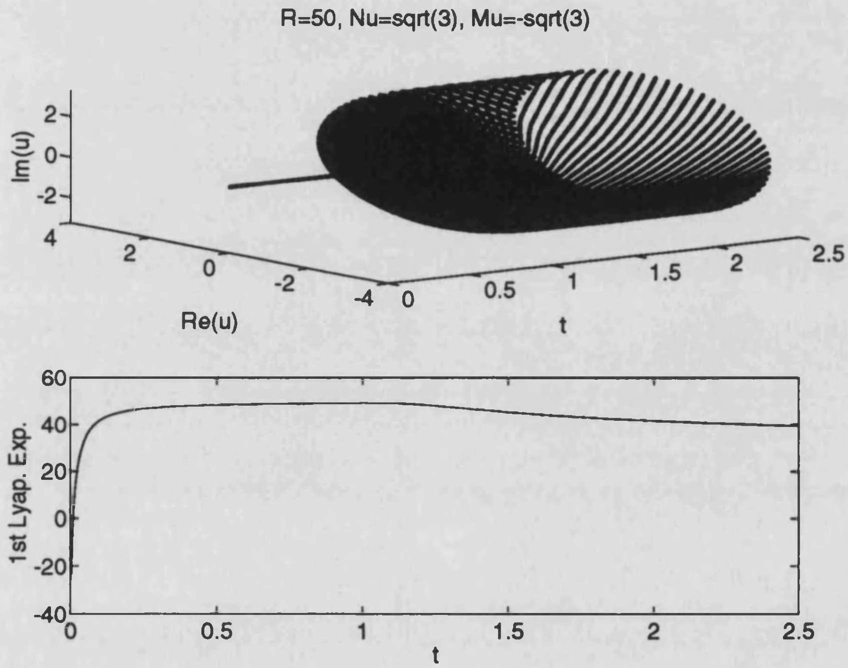

Figure 7.2: Solution and Lyapunov exponent for connection to unstable rotating wave, $R = 50$, $\nu = -\sqrt{3}$, $\mu = \sqrt{3}$.

In Figure 7.3 we have plotted the Fourier modes for the connection from the trivial solution to the first unstable rotating wave. This is for $R = 160$, $\nu = -\sqrt{3}$ and $\nu = \sqrt{3}$.

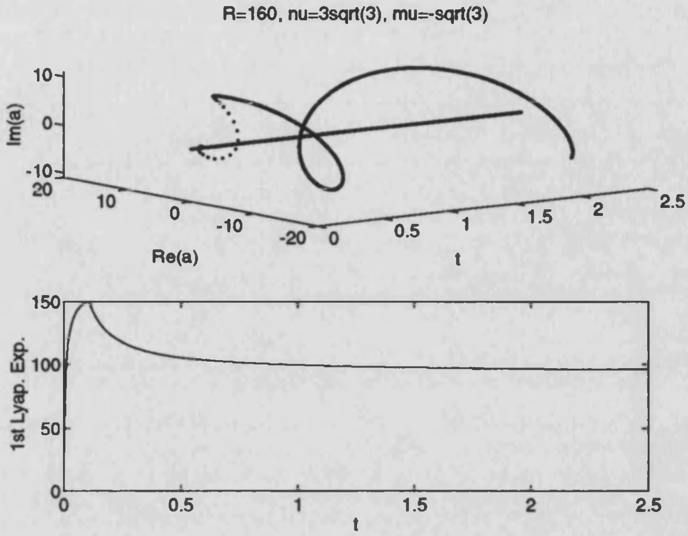

Figure 7.3: Connection to the first unstable rotating wave, $R = 160$, $\nu = -\sqrt{3}$ and $\nu = \sqrt{3}$.

For $R = 600$, $\nu = \sqrt{3}/2$ and $\mu = -\sqrt{3}$ we have plotted in Figure 7.4 the spatially homogenous wave reconstructed from the Fourier modes. By the linear stability analysis of section 3.5.4 the spatially homogenous wave is unstable for these parameter values and this is what we find numerically since the corresponding largest exponent is positive. The solution was found by perturbing the trivial solution in the correct Fourier modes, we note that with different initial data very different dynamical behaviour is observed (see for example 7.6).

In Figures 7.5 to 7.7 we have fixed $R = 160$, $\mu = -\sqrt{3}$ and also the initial condition: the trivial solution with a small pertubation in all the Fourier modes. In Figure 7.5 we see the Fourier modes for $\nu = \sqrt{3}/2$. For $\nu = 3\sqrt{3}$ we have plotted the Fourier modes against time in Figure 7.6 and in Figure 7.7 the corresponding largest Lyapunov exponent (top) and the real and imaginary parts of the reconstructed solution against time (below). The evolution in time is quite erratic and the largest Lyapunov exponent is positive which possibly indicates a chaotic solution. We note that this connection from the trivial solution to the stable possibly chaotic solution passes close to one of the unstable periodic orbits (for $0.5 \lesssim t \lesssim 1$).

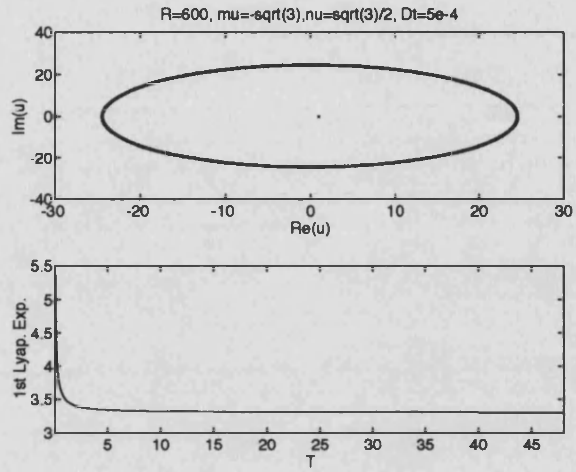

Figure 7.4: Spatially homogenous rotating wave and 1st Lyapuonv exponent for $R = 600$, $\nu = \sqrt{3}/2$ and $\mu = -\sqrt{3}$.

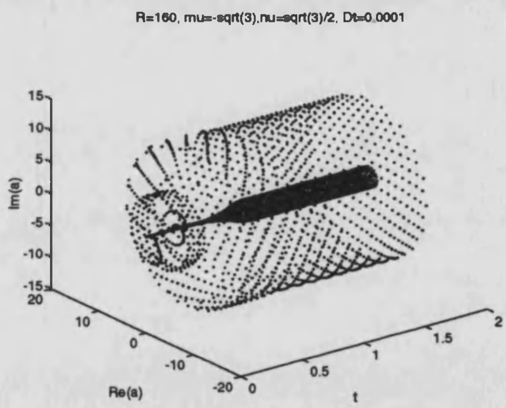

Figure 7.5: Fourier modes for $R = 160$, $\nu = \sqrt{3}/2$, and $\mu = -\sqrt{3}$.

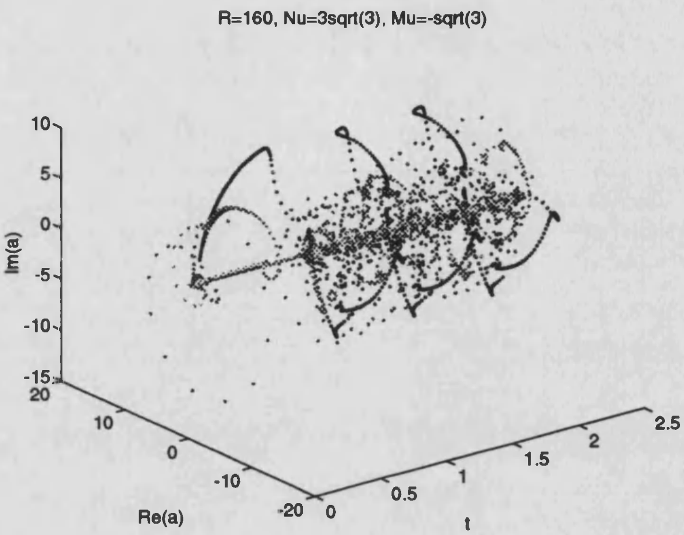

Figure 7.6: Fourier modes for $R = 160$, $\nu = 3\sqrt{3}$, and $\mu = \sqrt{3}$.

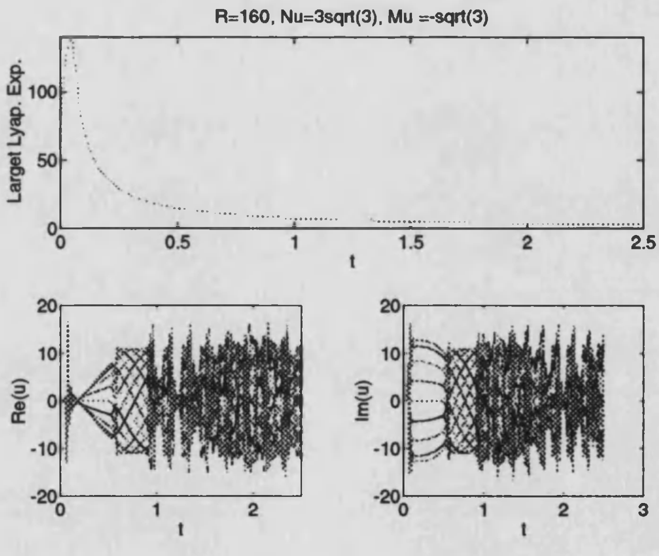

Figure 7.7: First Lyapunov exponent and solution for $R = 160$, $\nu = 3\sqrt{3}$, and $\mu = \sqrt{3}$.

7.4 Dirichlet Boundary Conditions

Consider the complex Ginzburg–Landau equation with Dirichlet boundary conditions on the interval $[0, 1]$.

$$U_t = RU - (1 + i\nu)A_d U - (1 + i\mu)|U|^2 U, \tag{7.4.17}$$

$$U^0 \in L^2 \text{ given;}$$

where

$$A_d := -\Delta \text{ with domain } \; D(A_d) := \left\{u \in L^2 : |A_d u|_{L^2} < \infty\right\}. \tag{7.4.18}$$

We now give the eigenvalues and vectors for the linear operator A_d.

Lemma 7.4.1 *The linear operator defined by* (7.4.18) *has eigenvalues* $\{\mu_m\}$,

$$\mu_m = m^2\pi^2 \quad m \in \mathbb{N},$$

and eigenvectors $\{\Phi_m\}$,

$$\Phi_m = \sin(m\pi x), \quad m \in \mathbb{N},$$

so that

$$A_d\Phi_m = \mu_m\Phi_m \quad \forall m \in \mathbb{N}.$$

Proof Straightforward.□

With this lemma in hand we define the spectral approximation in the Dirichlet case. For fixed $M \in \mathbb{N}$, $(M < \infty)$ we have the approximation

$$U(t) = \sum_{m=1}^{M} a_m(t)\Phi_m,$$

where the Fourier modes satisfy

$$\frac{d}{dt}a_m = Ra_m - (1 + i\nu)\mu_m a_m - \frac{1}{4}\sum_{\substack{j+k=\ell+m \\ j,k,\ell=1}}^{M} a_j a_k \overline{a_\ell} \quad \forall m = 1, \cdots, M. \tag{7.4.19}$$

We note that the results of sections 3.3–3.5 are true for (7.4.17) and in particular there exists an inertial manifold and solutions lie in a Gevrey class $\tau > 0$ and so again the spectral method is a natural way to discretize the equation. We note that for

Dirichlet boundary conditions we may prove using the Hopf Bifurcation Theorem that the trivial solution $U \equiv 0$ undergoes a Hopf bifurcation (see Theorem 7.4.1). However in this Dirichlet case we no longer have an explicit expression for the resulting periodic orbit (which no longer exists in a single Forier mode. Hence heuristically we may expect that more modes will be necessary for a good dynamic approximation than in the periodic boundary condition case.

Theorem 7.4.1 *The trivial solution $U \equiv 0$ to* (7.4.17) *undergoes a Hopf bifurcation at the points $R = m^2\pi^2$ for all $m \in \mathbb{N}$.*

Proof Straightforward after noting that we now have simple eigenvalues.□

We now review the work of Mischaikow and Morita on the Ginzburg—Landau equation with Dirichlet boundary conditions.

7.4.1 A Gradient Flow

Given $U^0 \in X$ re–consider the continuous dynamical system

$$U_t = F(U), \quad U(0) = U^0; \tag{7.4.20}$$

and the discrete dynamical system

$$U^{n+1} = F(U^n). \tag{7.4.21}$$

Definition 7.4.1 (Gradient) The dynamical system (7.4.20) or (7.4.21) is said to define a *gradient system* if $\exists I \in C(X, \mathbb{R})$ called a Lyapunov function such that all the following properties hold:

- $I(U) \geq 0 \; \forall U \in X$;
- $I(U) \to \infty$ as $\|U\| \to \infty$;
- $I(S(t)U^0)$ (or $I(S^nU^0)$) is non–decreasing in t (or n);
- If $I(S(t)U^0) = I(U^0)$ (or $I(S^nU^0) = I(U^0)$) then U^0 is an equilibrium for (7.4.20) (or (7.4.21)).

A common example of a continuous gradient system is the case when

$$F(U) = -\nabla I(U).$$

Recall that an equilibrium $\widetilde{U}$ for the continuous system (7.4.20) is said to be *hyperbolic* if none of the eigenvalues of $DF[\widetilde{U}]$ lie on the imaginary axis and for the discrete system (7.4.21) if none of the eigenvalues of $DF[\widetilde{U}]$ lie on the unit circle.

We also introduce the following notation for the set of all equilibria for (7.4.20) and (7.4.21)

$$\mathbf{C} \quad \mathcal{E} := \{U \in X : S(t)U = U \ \forall t > 0\};$$

$$\mathbf{D} \quad \mathcal{E} := \{U \in X : S^n U = U \quad \forall n \in \mathbb{N}\}.$$

Theorem 7.4.2 *Let (7.4.20) or (7.4.21) be a gradient system and assume that the Lyapunov function $I(U)$ satisfies the additional property that*

$$\exists \xi > 0 : V \in \mathcal{E} \implies I(V) \leq \xi.$$

Then there exists a global attractor $\mathcal{A}$ given by

$$\mathcal{A} = \{U^0 \in X : dist_X(U(t), \mathcal{E}) \to 0, t \to -\infty\}$$

and if $\mathcal{E}$ comprises only hyperbolic equilibrium points then

$$\mathcal{A} = \bigcup_{V \in \mathcal{E}} W^u(V).$$

Proof See [63] and for related results [66].□

Re–consider the complex Ginzburg–Landau equation with Dirichlet boundary conditions as given by equation (7.4.17). Then we re–scale in space and time as follows.

Let $p \in \mathbb{R}$ be defined by $p^2 := R > 0$ and divide equation (7.4.17) by p^3. This yields

$$\frac{1}{p^3}U_t = \frac{U}{p} - \frac{1}{p^3}(1+i\nu)A_d U + (1+i\mu)\left|\frac{U}{p}\right|^2 \frac{U}{p}.$$

Now define $V(x,t) \in \mathbb{C}$ by $V(x,t) := U(x,3t)/p$, then $V(x,t)$ satisfies

$$V_t = V - \frac{1}{p^2}(1+i\nu)A_d V - (1+i\mu)|V|^2 V.$$

Re–labeling so that $\nu = \kappa$ and $\nu = 1/R$ we get

$$V_t = -\nu(1 + i\kappa)A_d V + V - (1 + i\mu)|V|^2 V. \tag{7.4.22}$$

and the bifuraction parameter corresponding to R for this equation is $1/\nu$.

Equation (7.4.22) is the form of the Ginzburg–Landau equation considered by Mischaikow and Morita [97]. We now review that part of their work which shows that for $\mu \approx \kappa$ the system (7.4.22) is a gradient system.

Let $\mu = \kappa$ in equation (7.4.22) and transform variables again by defining

$$W(x,t) := e^{i\kappa t} V(x,t).$$

Then W satisfies

$$\begin{aligned} W_t &= i\kappa W + e^{i\kappa t} V_t \\ &= (1 + i\kappa)\left[\nu W_{xx} + W - |W|^2 W\right]. \end{aligned} \tag{7.4.23}$$

For $\kappa = 0$ this is the complex Chafee-Infante equation considered, for example, in [10, 7] or [20] in which it was shown that this system was gradient. We now consider the case $\kappa \neq 0$.

Lemma 7.4.2 *There exists a Lyapunov function $I : X \to \mathbb{R}$ for the system* (7.4.23) *given by*

$$I(W) = (1 + \kappa^2)\int_0^1 \left\{\nu |W_x|^2 - (|W|^2 - \frac{1}{2}|W|^4)\right\} dx;$$

and hence the system is a gradient system.

Proof See Mischaikow and Morita [97].□

This may be applied to find a Morse decomposition for (7.4.23) and so for (7.4.22). Lemma 7.4.2 is extended in [97] to give.

Corollary 7.4.1 *There exists $\epsilon > 0$ such that for $|\kappa - \mu| < \epsilon$ the system* (7.4.22) *has the same Morse decomposition as $\kappa = \mu$.*

Proof The extension from $\kappa = \mu$ is inferred in [97] by noting that their analysis is stable under perturbations. □

7.4.2 Numerical results

For the Ginzburg–Landau equation with Dirichlet boundary conditions we computed with the equation in the form of (7.4.22). For the equation in this form we note that when $\kappa = \mu$ the stationary solutions of (7.4.23) correspond to periodic solutions of (7.4.22) with period $\kappa \bmod 2\pi$.

In Figure 7.8 we present for $\kappa = \mu = 1$ the bifurcation digram from the trivial solution in the bifurcation parameter $1/\nu$. This was found using the numerical continuation code AUTO [37] for a 20 mode Galerkin approximation. Each branch from the trivial solution represents a periodic solution of period 2π as expected. We have used "o" to denote that the branch is stable. Thus the first branch is stable and the other branches are unstable. We have indicated on the Figure the dimensions of the unstable and center manifolds for the unstable solutions in the following manner $(x, y) := (\dim(W^U(U)), \dim W^c(U))$. The center manifolds of the unstable branches correspond to those solutions being periodic.

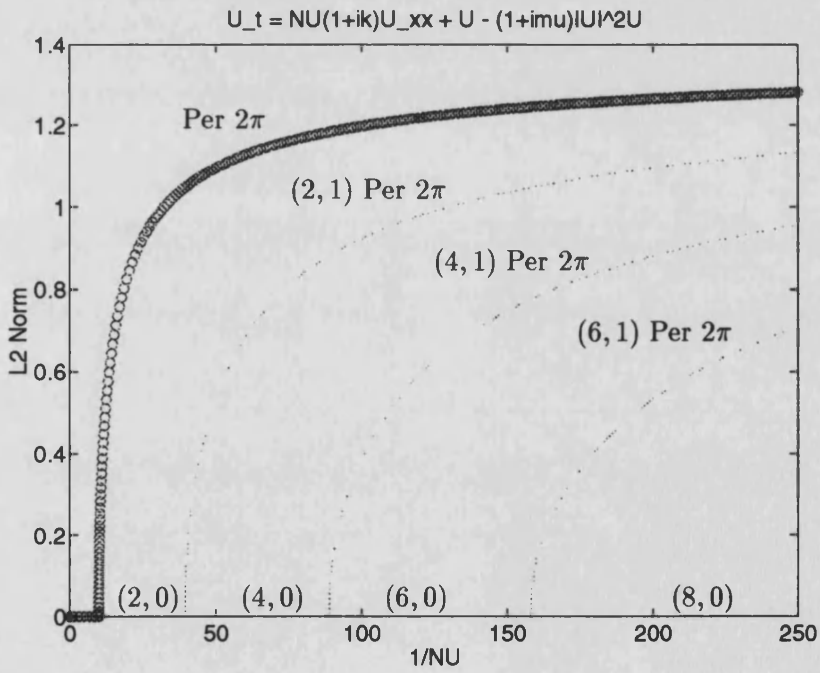

Figure 7.8: Bifurcation diagram for $\kappa = \mu = 1$.

We may easily find heteroclinic connections to the stable periodic orbit by an initial value solver. For connections to the unstable solutions this is not evident as we do not know a priori the structure of the periodic orbits (unlike for the periodic boundary conditions case). However examining the periodic orbits we find they have a similar stucture in Fourier space as that found for the Chaffee–Infante, see Henry [69] and Bai [3]. This structure allows us to perturb the trivial solution in certain Fourier modes to remain in the subspace associated with the connection. For example the stable branch has odd non–zero Fourier modes and the first unstable branch only has certain even non–zero Fourier modes. Perturbing the trivial solution in the correct Fourier modes yields the connection to the first unstable branch. This connection is shown in Figure 7.9 for $1/\nu = 120$, $\kappa = 1$ and $\mu = 1$ in Fourier space and in Figure 7.10 we have plotted the solution reconstructed from the Fourier modes. Figure 7.11 shows the solution for the same values of ν, κ and μ, in *a.* we have plotted the solution projected in time onto the complex plane, in *b.* the evolution of the point $U(1/2)$ is plotted in time, in *c.* we have plotted $\mathrm{Re}(U(1/2))$ against time and in *d.* $\mathrm{Im}(U(1/2))$ against time.

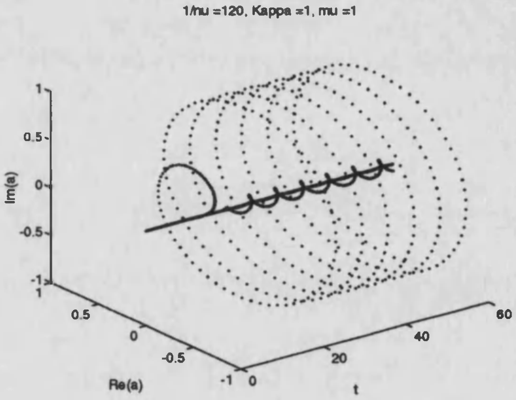

Figure 7.9: Connection from $U \equiv 0$ to first unstable branch of periodic orbits in Fourier space for $1/\nu = 120$, $\kappa = 1$ and $\mu = 1$.

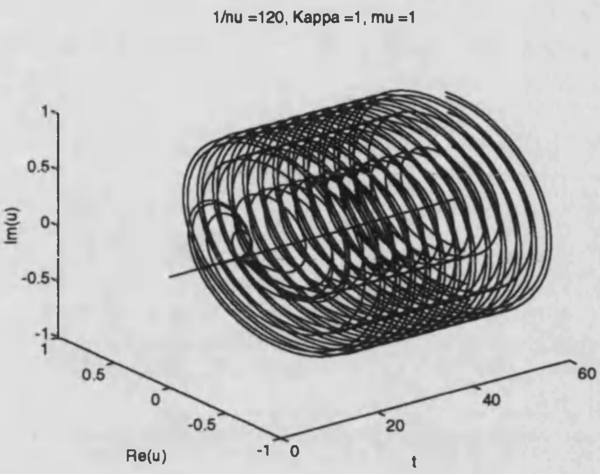

Figure 7.10: Solution connecting from $U \equiv 0$ to first unstable branch of periodic orbits for $1/\nu = 120$.

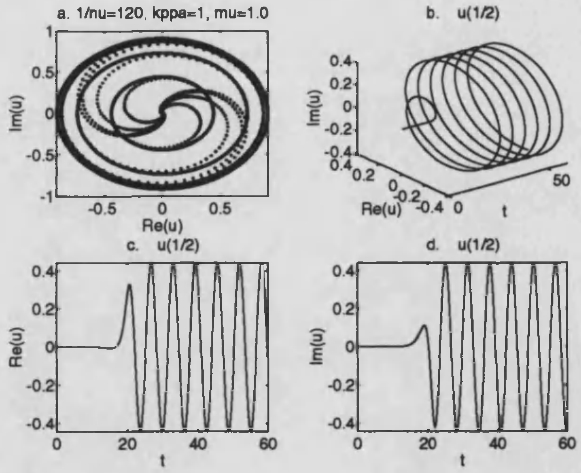

Figure 7.11: The solution from $U \equiv 0$ to first unstable branch of periodic orbits for $1/\nu = 120$.

These results are all for the case $\kappa = \mu = 1$ which by Lemma (7.4.2) transforms to a gradient system and hence by Theorem (7.4.2) has a global attractor which is well understood.

Our next step was to do continuation in the parameter μ to follow the periodic orbits away from the gradient case. This was accomplished using the continuation code AUTO [37] and the results may be seen in Figures 7.12 and 7.13. The stable branch is plotted with a $\circ$ and $-$ whereas the unstable branches have been plotted with a dashed–dotted line.

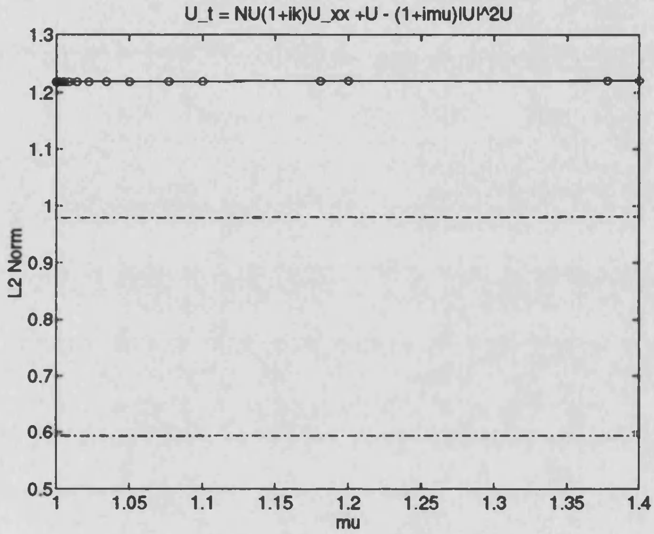

Figure 7.12: Following in μ the periodic solutions away from $\kappa = \mu = 1$.

Figure 7.12 shows the branches of periodic orbits in $(||U||, \mu)$ plane and Figure 7.13 the branches in the (period,μ) plane. Clearly as we move away from the gradient case the periodic solutions begin to rotate with different periods. The stable branch remains stable and the dimension of the unstable manifold of the second branch remains constant. However for the first unstable branch the dimension of the unstable manifold changes at $\mu = 1.4$ from 2 to 1. This is either a true phenomena or a numerical artifact and warrants further investigation.

The stucture of the periodic solutions in Fourier spaces persists as μ is varied and so we may again compute heteroclinic connections to unstable orbits using an initial

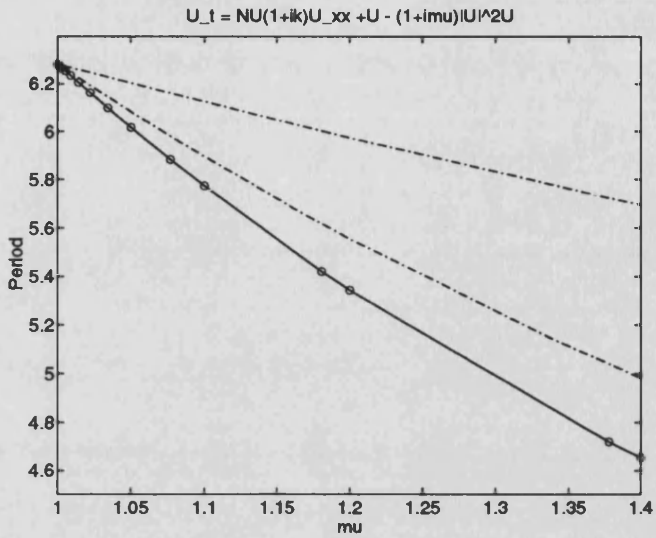

Figure 7.13: Periodic orbits followed in μ away from $\kappa = \mu = 1$.

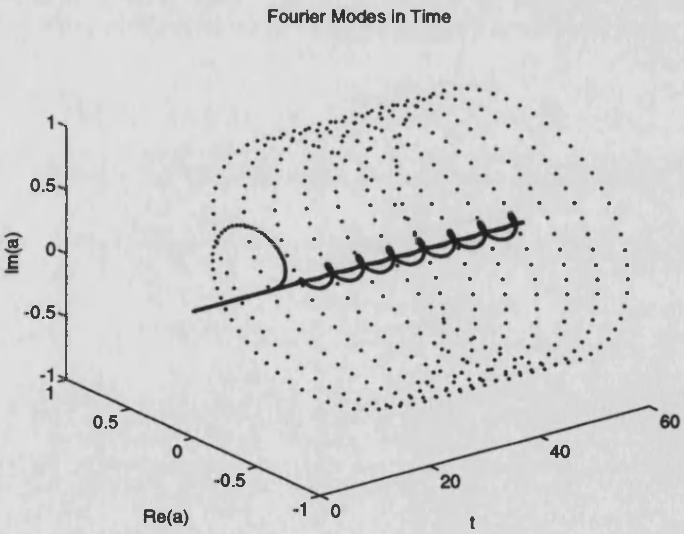

Figure 7.14: Connection in Fourier space from $U \equiv 0$ to the first unstable branch for $1/\nu = 120$, $\kappa = 1$ and $\mu = 1.2$.

value solver. The result of this computation may be seen in Fourier space in Figure 7.14 where we have computed the connection from the trivial solution to the first unstable branch for $1/\nu = 120$, $\kappa = 1$ and $\mu = 1.2$. This should be compared with the connection found for $\mu = 1$ (Fig 7.10). In Figure 7.15 *a.* we have plotted the corresponding solution projected in time onto the complex plane, in *b.* we have plotted the evolution of the point $U(1/2)$, in *c.* we have plotted $\mathrm{Re}(U(1/2))$ against time and in *d.* the $\mathrm{Im}(U(1/2))$ against time. We note that essentially the only difference we can observe between Figures 7.10 and 7.15 is that the period and phase are different.

Counting the dimensionality of the unstable and stable subspaces of the trivial solutions and periodic solutions respectively we see that we may expect a non–trivial manifold from the trivial solution to the periodic orbit of the first branch. For the Chaffee–Infante equation [3] use numerical continuation to explore the non–trivial manifold of heteroclinic connections from the trivial branch to the first branch of unstable steady state solutions. Adapting code donated by Dr Bai which is discussed in [4] we are currently examining the nature of the manifold of connections from the trivial solution to the first unstable branch.

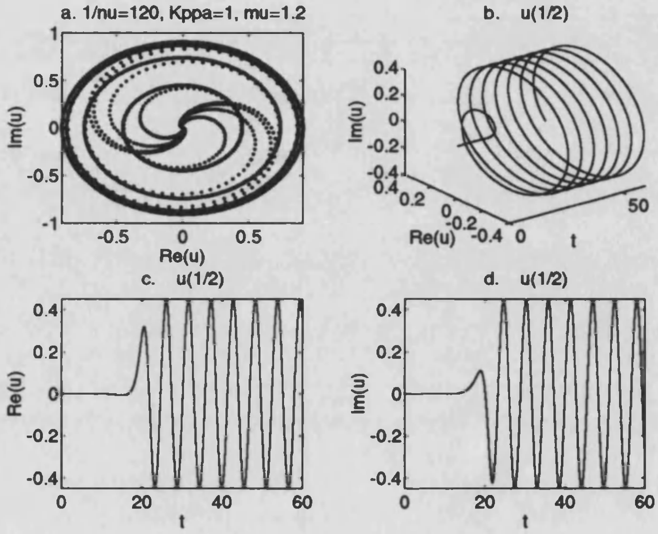

Figure 7.15: Connection from $U \equiv 0$ to the first unstable branch for $1/\nu = 120$, $\kappa = 1$ and $\mu = 1.2$.

Bibliography

[1] R.A. Adams. *Sobolev Spaces.* Academic Press, New York, 1975.

[2] S. Allen and J.W. Cahn. A microscopic theory for antiphase boundary motion and its application to antiphase domain coarsing. *Acta. Metall.*, 27:1084–1095, 1979.

[3] F. Bai, A. Spence, and A. M. Stuart. The numerical computation of heteroclinic connections in systems of gradient partial differential equations. *SIAM J. Appl. Math.*, 53:743–769, 1993.

[4] F. Bai, A. Spence, and A. M. Stuart. Numerical computation of connecting orbits in discrete and continuous dynamical systems. 1994. In Preparation.

[5] D.L. Barrow and P.W. Bates. *Bifurcation of Periodic Travelling Waves for a Reaction-Diffusion System.* 964. Springer-Verlag, New York, 1982.

[6] D.L. Barrow and P.W. Bates. Bifurcation and stability of periodic travelling waves for a reaction–diffusion system. *Journal of Diffential Equations*, 50:218–233, 1983.

[7] D.L. Barrow and P.W. Bates. Bifurcation from collinear solutions to a reaction-diffusion system. *Contemporary Mathematics*, 17:179–187, 1983.

[8] M. Bartuccelli, P. Constantin, C.R. Doering, J.D. Gibbon, and M. Gisselfält. On the possibility of soft and hard turbulance in the complex Ginzburg–Landau equation. *Physica D*, 44:421–444, 1990.

[9] M. Bartuccelli, C.R. Doering, and J.D. Gibbon. Ladder theorems for the 2d and 3d Navier–Stokes equations on a finite periodic domain. *Nonlinearity*, 4:531–542, 1991.

[10] P.W. Bates. Invariant manifolds for perturbations of non-linear parabolic systems with symmetry (preliminary report). *Lectures in Applied Mathematics*, 23:209–217, 1986.

[11] G. Benettin, L. Galgani, A. Giorgilli, and J-M. Strelcyn. Lyapunov characteristic exponents for smooth dynamical systems and for Hamiltonian systems; a method for computing all of them. Part1: Theory. *Meccanica*, 15:9–20, 1980.

[12] G. Benettin, L. Galgani, A. Giorgilli, and J-M. Strelcyn. Lyapunov characteristic exponents for smooth dynamical systems and for Hamiltonian systems; a method for computing all of them. Part2: Numerical application. *Meccanica*, 15:21–31, 1980.

[13] W.-J. Beyn. The numerical computation of connecting orbits in dynamical systems. *IMA Journal of Numerical Analysis*, 9:379–405, 1990.

[14] W.-J. Beyn. Numerical analysis of homoclinic orbits emanating from a Takens–Bogdanov point. Technical Report 92–065, Universität Bielefeld, 1992.

[15] J.E. Billotti and J.P. LaSalle. Dissipative periodic processes. *Bull. A. M.S*, 77:1082–1089, 1971.

[16] N. Bourbaki. *Espaces Vectoriels Topologiques.* Masson, Paris, 1981.

[17] J.H. Bramble, A.H. Schatz, Thomée V., and L.B. Wahlbin. Some convergence estimates for semi–discrete Galerkin type approximations for parabolic equations. *SIAM J. Numer. Anal.*, 14:218–241, 1997.

[18] A. Brandstater, J. Swift, H.L. Swinney, A. Wolf, J.D. Farmer, E. Jen, and P.J. Crutchfield. Low dimensional chaos in a hydrodynamic system. *The American Physical Society*, 51(16):1442–1445, October 1983.

[19] P. Brenner, V. Thomée, and L. Wahlbin. *Besov Spaces and Applications to Difference Methods for Initial Value Problems.* Springer-Verlag, New York, 1975.

[20] K.J. Brown, P.C. Dunne, and R.A. Gardner. A semilinear parabolic system arising in the theory of superconductivity. *Journal of Differential Equations*, 40:232–252, 1981.

[21] J.W. Cahn and J.E. Hilliard. Free energy of a non-uniform system i. interfacial free energy. *J. Chem. Phys.*, 28:258–267, 1958.

[22] J. Carr. *Center Manifolds.* Springer–Verlag, 1993.

[23] N. Chafee and E. Infante. A bifurcation problem for a nonlinear partial differential equation of parabolic type. *SIAM J. Appl. Anal.*, 4:17–37, 1974.

[24] A.R. Champneys and Yu.A. Kuznetsov. Numerical detection and continuation of codimension two homoclinic bifurcations. Technical Report AM-R9308, Centruum voor Wiskunde en Informatica, Sept. 1993.

[25] L. Chua and T. Parker. *Practical Numerical Algorithms for Chaotic Systems.* Springer–Verlag, New York., 1989.

[26] P. Collet and J-P. Eckmann. *Iterated Maps on the Interval as Dynamical Systems.* Progress in Physics. Birkhauser, Boston, 1980.

[27] P. Constantin and C. Foias. Global Lyapunov exponents, Kaplan–Yorke formulas and the dimension of the attractors for 2D Navier–Stokes equations. *Communications on Pure and Applied Mathematics*, 38:1–27, 1985.

[28] P. Constantin and C. Foias. *Navier–Stokes Equations.* University of Chicago Press, Chicago, 1988.

[29] P. Constantin, C Foias, B. Nicolaenko, and R. Temam. *Integral Manifolds and Inertial Manifolds for Dissipative Partial Differential Equations.* Applied Mathematical Sciences *70.* Springer–Verlag, 1989.

[30] P. Constantin, C Foias, and R. Temam. Attractors representing turbulent flows. *Mem. Amer. Math. Soc*, 53(314), 1985.

[31] J. Crutchfield and N. Packard. Symbolic dynamics and noisy chaos. *Physica D*, 7:201–223, 1983.

[32] R. Dautray and J-L. Lions. *Mathematical Analysis and Numerical Methods for Science and Technology*, volume 2, Functional and Variational Methods. Springer–Verlag, New York, 1985.

[33] F. Demengel and J-M. Ghidaglia. Nonlinear evolution partial differential equations. *C.R. Acad. Sci.*, 307:453–458, 1988. Series:I, Math.Analysis/Numerical Analysis.

[34] R. Devaney. *An Introduction to Chaotic Dynamical Systems.* Addison-Wesley, second edition, 1989.

[35] L. Dieci, R.D. Russell, and E.S. Van Vleck. Unitary integrators and applications to continuous orthonormalization techniques. *SIAM J. Numerical Analysis*, 1992.

[36] L. Dieci, R.D. Russell, and E.S. Van Vleck. On the computation of Lyapunov exponents for continuous dynamical systems. Technical report, Georgia Tech /Simon Fraser University/Simon Fraser University, 1993.

[37] E. Doedel. *AUTO 86 User Manual Software for Continuation and Bifurcation Problems in Ordinary Differential Equations*, May 1986. VAX/VMS.

[38] E. Doedel and M.J. Friedman. Numerical computation of heteroclinic orbits. *J. Comp. Appl. Math.*, 26:159–170, 1989.

[39] A. Doelman. Finite dimensional models of the Ginzburg-Landau equation. *Nonlinearity*, 4:231–250, 1990.

[40] A. Doelman and E.S. Titi. Regularity of solutions and the convergence of the Galerkin method in the Ginzburg–Landau equation. *Numerical Functional Analysis and Optimization*, Preprint. Submitted Nov.1992, Dept. Mathematics, Universty Utrecht.

[41] C. Doering, J. Gibbon, D. Holm, and B. Nicolaenko. Low-dimensional behaviour in the complex Ginzburg-Landau equation. *Nonlinearity*, 1:279–309, 1988.

[42] J. Duan, E.S. Titi, and P. Holmes. Regularity, approximation and asymptotic dynamics for a generalised Ginzburg–Landau equation. *Nonlinearity*, Submitted.

[43] N. Dunford and J.T. Schwartz. *Linear Operators Part 1: General Theory.* Wiley-Interscience, New York, 1958.

[44] J.P. Eckmann and D. Ruelle. Ergodic theory of chaos and strange attractors. *Reviews of Modern Physics*, 57(3 Part 1):617–656, July 1985.

[45] C.M. Elliott and S. Larssen. Error estimates with smooth and non-smooth data for a finite element method for the Cahn–Hilliard equation. *Math. Comp*, 49:359–377, 1987.

[46] C.M. Elliott and A. M. Stuart. The global dynamics of discrete semilinear parabolic equations. *SIAM J. Numer. Anal.*, 30(6), 1993.

[47] C. Foias, M. Jolly, I. Kevrekidis, and E. Titi. Dissipitivity of numerical schemes. *Nonlinearity*, 4:591–613, 1991.

[48] C. Foias, G. Sell, and R. Temam. Inertial manifolds for nonlinear evolutionary equations. *Journal of Differential Equations*, 73:309–353, 1988.

[49] C. Foias and R. Temam. Gevrey class regularity for the solutions of the Navier-Stokes equations. *Journal of Functional Analysis*, 87:359–369, 1989.

[50] C. Foias and E. Titi. Determining nodes, finite difference schemes and inertial manifolds. *Nonlinearity*, 4:135–153, 1991.

[51] A. Friedman. *Partial Differential Equations.* Holt, Rinehart and Winston, Inc, New York, 1969.

[52] H. Fujita and H. Mizutani. On the finite element method for parabolic equations. *J. Math. Soc. Japan*, 28:749–771, 1976.

[53] A. Gardiner and A. Mitchell. Periodic bifurcations of an explicit discretization of the Ginzburg–Landau equation. Numerical Analysis Report NA/137, Dundee, Jan 1992.

[54] M. Gevrey. *Ouevres de Maurice Gevrey.* Editions du Centre national de la recherche scientifique, Paris, 1970.

[55] J.M. Ghidaglia and B. Héron. Dimension of the attractors associated to the Ginzburg–Landau partial differential equation. *Physica 28D*, 1987.

[56] J.M. Ghidaglia and R. Temam. Attractors for damped nonlinear hyperbolic equations. *J. Math. Pures Appl.*, 66, 1987.

[57] I. Goldhirsch, P-L. Sulem, and S. Orszag. Stability and Lyapunov stability of dynamical systems: A differential approach and a numerical method. *Physica D*, 27:311–337, 1987.

[58] D. Gottlieb and S.A. Orszag. *Numerical Analysis of Specral Methods Theory and Applications.* CBMS–NSF. SIAM, 1977. Regional Conference Series in Applied Mathematics.

[59] J. Guckenheimer and P. Holmes. *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields.* Applied Mathematical Sciences *42*. Springer-Verlag, New York, 1983.

[60] W. Hackbusch. *Elliptic Differential Equations, Theory and Numerical Treatment.* Computational Mathematics, *18*. Springer-Verlag, Berlin, 1987.

[61] T. Hagstrom and H.B. Keller. Exact boundary conditions at an artificial boundary for partial differential equations in cylinders. *SIAM J. Math. Anal.*, 17:322–341, 1986.

[62] T. Hagstrom and H.B. Keller. Asymptotic boundary conditions and numerical methods for nonlinear elliptic problems on unbounded domains. *Math. Comp.*, 48:449–470, 1987.

[63] J.K. Hale. *Asymptotic Behaviour of Dissipative Systems.* A. M.S., New York, 1988.

[64] J.K. Hale, X-B. Lin, and G. Raugel. Upper semicontinuity of attractors for approximations of semigroups and partial differential equations. *Mathematics of Computation*, 50(181):89–123, 1988.

[65] J.K. Hale and G. Raugel. Lower semicontinuity of attractors of gradient systems and applications. *Annali di Matematica pura ed applicata*, 4:281–326, 1989.

[66] J.K. Hale and G. Raugel. Convergence in gradient-like systems with applications to PDE. *Z Angew Math Phys*, 43:63–124, 1992.

[67] P. Hartman. *Ordinary Differential Equations.* Birkhauser–Verlag, Boston, 1982.

[68] D. Henry. *Geometric Theory of Semilinear Parabolic Equations*, volume 840 of *Lecture Notes in Mathematics.* Springer – Verlag, Berlin, Heidelberg, 1981.

[69] D. Henry. Some infinite dimensional Morse–Smale systems defined by parabolic differential equations. *J. Diff. Equ.*, 59, 1985.

[70] J.G. Heywood and R. Rannacher. On the question of turbulence modeling by approximate inertial manifolds and the non–linear Galerkin method. *SIAM J. Numer. Anal*, 1992.

[71] A.T. Hill. *Attractors for Convection–Diffusion Equations and their Numerical Approximation.* PhD thesis, Oxford University, 1993.

[72] A.T. Hill and E. Süli. Set convergence for discretizations of the attractor. Technical Report 93/8, Oxford, June 1993.

[73] A.T. Hill and E. Süli. Upper semicontinuity of attractors for linear multistep methods approximating sectorial evolution equations. Technical Report 92/8, Oxford, May 1993.

[74] M. Hirsch and S. Smale. *Differential Equations, Dynamical Systems and Linear Algebra.* Academic Press inc., New York, 1974.

[75] G. Huber and P. Alstrom. Universal decay of vortex density in two dimensions. *Physica A*, 195:448–456, 1993.

[76] G. Huber, P. Alstrom, and T. Bohr. Nucleation and transients at the onset of vortex turbulence. *Physical Review Letters*, 69 No.16:448–456, 1992.

[77] A.R. Humphries. Spurious solutions of numerical methods for initial value problems. *IMA Journal of Numerical Analysis*, 1992.

[78] A.R. Humphries. Approximation of attractors and invariant sets by Runge–Kutta methods. 1994. In Preparation.

[79] A.R. Humphries and A. M. Stuart. Approximation of dissipative partial differential equations over long time intervals. In *Numerical Analysis, Dundee, 1993*. Longman, New York, 1994. In preparation.

[80] A.R. Humphries and A. M. Stuart. Runge-Kutta methods for dissipative and gradient dynamical systems. *SIAM J. Num. Anal.*, 1994. Submitted.

[81] F. John. *Partial Differential Equations*. Applied Mathematical Sciences. Springer–Verlag, fourth edition, 1982.

[82] D.A. Jones, L.G. Margolin, and E.S. Titi. On the effectiveness of the approximate inertial manifold – a computational study. *J. of Theoretical and Computational Fluid Mechanics*, To Appear.

[83] D.A. Jones and A. M. Stuart. Attractive invariant manifolds under approximation. In Preparation.

[84] L. Keefe. Dynamics of the peturbed wavetrain solutions to the Ginzburg-Landau equation. *Stud. Applied Maths*, 73:91–153, 1985.

[85] H. Keller. *Numerical Solutions of Two point Boundary Value Problems*, volume 24 of *CBMS–NSF*. SIAM, 1976. Regional Conference Series in Applied Mathematics.

[86] U. Kirchgraber and D. Stoffer. On the definition of Chaos. *Z.angew.Math.Mech.*, 69(7):175–185, 1989.

[87] I. Kukavica. On the number of determining nodes for the Ginzburg–Landau equation. *Nonlinearity*, 5:997–1006, 1992.

[88] O. Ladyzhenskaya. *Attractors for Semigroups and Evolution Equations*. CUP, 1991.

[89] S. Larsson. Nonsmooth data error estimates with applications to the study of the long–time behavior of finite element solutions of semilinear parabolic problems. *IMA Journal of Numerical Analysis*, To Appear.

[90] Y. Liu, L. Liu, and T. Tang. The numerical computation of connecting orbits in dynamical systems: a rational spectral approach. Technical Report 92–17, Simon Fraser University, 1992.

[91] Y. Liu and R.D. Russell. Linear system solvers for boundary value odes. *J. Comput. and Appld. Math.*, 45:103–117, 1993.

[92] E. Lorenz. Deterministic non-periodic flow. *Journal of Atmos. Sciences*, 20:130–141, March 1963.

[93] A. Lyapunov. *Problém Géneral de la Stabilité du Mouvement.* Number 17 in Annals of Mathematics Studies. Princeton University Press, 1949.

[94] M. Marion and R. Temam. Nonlinear Galerkin methods. *SIAM J. Num. Anal.*, 26:1139–1157, 1990.

[95] R. Mickens. *Difference Equations.* Van Nostrand Rheihold Company Inc., 1987.

[96] K. Mischaikow. A c–graph approach for studing the dynamics of a system of parabolic equations. December 1989.

[97] K. Mischaikow and Y. Morita. Dynamics on the global attractor of a gradient flow arising from the Ginzburg–Landau equation. Technical Report CDSNS92-100, Georgia Institute of Technology / Ryukoku University, 1992.

[98] K. Mischaikow and M. Mrozek. Chaos in the Lorenz equations: A computer assisted proof. Technical Report CDSNS93-124, Georgia Institute of Technology - Center for Dynamical Systems and Nonlinear Studies, November 1993.

[99] Friedman M.J. and E.J. Doedel. Computational methods for global analysis of homoclinic and heteroclinic orbits: a case study. *J. Diff. Equ.*, 1994. To Appear.

[100] J.I. Mokin. A mesh analogue of the imbedding theorem for type w classes. *Journal Numerical Mathematics*, 6:1–16, 1971.

[101] D. Nychka, D. McCaffrey, S. Ellner, and A. Gallant. Estimating Lyapunov exponents with nonparametric regression. *Inst. Stat. Mime.*, 1990.

[102] M. Oliver. Attractors, regularity and length scales in the complex Ginzburg–Landau equation with a nonlinearity of arbitrary order. Master's thesis, Imperial College, London, December 1992.

[103] V.I. Oseledec. A multiplicative ergodic theorem. Ljapunov characteristic numbers for dynamical systems. *Trans.Moscow Math.Soc.*, 19:197–231, 1968.

[104] A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations.* Applied Mathematical Sciences *44*. Springer-Verlag, New York, 1983.

[105] K. Promislow. Induced trajectories and approximate inertial manifolds for the Ginzburg-Landau equation. *Physica D*, 41:232–252, 1990.

[106] J-P. Ramis. Théorèmes d'indices Gevrey pour les Équations Différentielles Ordinaires. *Memoirs of the Amer. Math. Soc.*, 48(296):232–252, 1984.

[107] W. Rheinboldt and J. Burhardt. **Pitcon**: University of Pittsburgh Continuation Program. University of Pittsburgh, Pittsburgh, PA 15260.

[108] R.D. Richtmyer and K.W. Morton. *Difference Methods for Initial-Value Problems.* Interscience Tracts in Pure and Applied Mathematics *4*. John Wiley and Sons, New York, second edition, 1967.

[109] F. Riesz and B. Sz.-Nagy. *Functional Analysis.* Dover, New York, 1955 Republished 1990.

[110] J. Rodriguez and L. Sirovich. Low-dimensional dynamics for the Ginzburg-Landau equation. *Physica D*, 43:77–86, 1990.

[111] J. Rodriguez, L. Sirovich, and B. Knight. Two boundary value problems for the Ginzburg-Landau equation. *Physica D*, 43:63–76, 1990.

[112] D. Ruelle. Sensitive dependence on initial condition and turbulent behavior of dynamical systems. *ANALS of The New York Academy of Sciences*, 316.

[113] D. Ruelle. Characteristic exponents and invariant manifolds in Hilbert space. *Ann. Math.*, 115(243), 1982.

[114] D. Ruelle. *Chaotic Evolution and Strange Attractors.* CUP, 1987.

[115] M. Sano and Y. Sawada. Measurement of the Lyapunov spectrum from a chaotic time series. *Phys. Rev. Lett.*, 55:1082–1085, 1985.

[116] G. Sansone and R. Conti. *Non–Linear Differential Equations.* Pergamon Press, Oxford, 1964.

[117] J. Sanz-Serna. Two topics in nonlinear stability. July 1990. SERC Summer School, Lancaster.

[118] J. Sanz-Serna and J. Lopez-Marcos. A definiton of stability for nonlinear problems. *Numerical Treatment of Differential Equations*, 1991.

[119] J. Shen. Convergence of approximate attractors for a fully discrete system for reaction-diffusion equations. *Numer. Funct. Anal. and Optiz.*, 10:1213–1234, 1989.

[120] I. Shimada and T. Nagashima. A numerical approach to ergodic problem of dissipative dynamical systems. *Prog. Theor. Phys.*, 61:1605–1616, AUG 1979.

[121] G.D. Smith. *Numerical Solution of Partial Differential Equations : Finite Difference Methods.* Oxford Applied Mathematics and Computing Science Series. OUP, third edition, 1985.

[122] C. Sparrow. *The Lorenz Equations :Bifurcations, Chaos and Strange Attractors.* Applied Mathematical Sciences *41.* Springer-Verlag, New York, 1982.

[123] K. Stewartson and J.T. Stuart. A non–linear instability theory for a wave system in a plane Poiseuille flow. *J. Fluid Mech*, 48:529–545, 1971.

[124] J.C. Strikwerda. *Finite Difference Schemes and Partial Differential Equations.* Mathematics Series. Wadsworth and Brooks/Cole Advanced Books and Software, Pacific Grove, California, 1989.

[125] A. M. Stuart. Numerical analysis of dynamical systems. *Acta Numerica*, 1994. Submitted.

[126] E. Süli, 1992. Personal Communication.

[127] P. Takáč. Invariant 2-tori in the time-dependent Ginzburg-Landau equation. *SIAM*, 1992.

[128] R. Temam. *Infinite-Dimensional Dynamical Systems in Mechanics and Physics.* Applied Mathematical Sciences *68.* Springer-Verlag, New York, 1988.

[129] T.M. Vaneck. *Finding Regular and Chaotic Responses in Non-Linear Mechanical Systems.* PhD thesis, Stanford University, August 1993.

[130] L. Wahlbin. Finite element methods for evolution equations. In *Finite Element Methods for Evolution Equations.* SERC Numerical Analysis Summer School, July 1990. Lancaster University.

[131] A. Wathen. Realistic bounds for the Galerkin mass matrix. *IMA Journal Numerical Analysis*, 7:449–457, 1987.

[132] W. Wedig. Lyapunov exponents and invariant measures of dynamical systems. *ISNM*, 97:367–376, 1991.

[133] S. Wiggins. *Introduction to Applied Nonlinear Dynamical Systems and Chaos.* Number 2 in Text in Applied Mathematics. Springer–Verlag, New York, 1990.

[134] A. Wolf. Quantifying chaos with Lyapunov exponents. In A. v. Holden, editor, *CHAOS*, chapter 13. Manchester University Press, 1986.

[135] A. Wolf, J. Swift, L. Swinney, and J. Vastano. Determing Lyapunov exponents from a time series. *Physica D*, 16:285–317, 1985.

[136] Y. Yan. Dimensions of attractors for discretizations for Navier–Stokes equations. *Journal of Dynamics and Differential Equations*, 4(2):275–340, 1992.

[137] Y. Yan. Attractors and error estimates for discretizations of incompressible Navier–Stokes equations. *SIAM Journal Numerical Analysis*, Submitted. Abbreviated Title: Attractors and Error Estimates.

[138] Y. Yan. Attractors and dimensions for discretizations of a weakly damped Schrödinger equation and a Sine–Gorden equation. *Nonlinear Analysis, TMA*, To Appear.

[139] Y. Yang. Computation, dimensionality, and zero–dissipation limit of the Ginzburg–Landau wave equation. *IMA Journal of Applied Mathematics*, 45:175–194, 1990.

[140] Y. Yang. Convergence of the pseudospectral method for the Ginzburg–Landau equation. *Journal of Mathematical Analysis and Applications*, 147:556–568, 1990.

[141] K. Yosida. *Functional Analysis*. Springer–Verlag, New York, second edition, 1968.

[142] Z. Yulin. *Applications of Discrete Functional Analysis to the Finite Difference Method*. International Academic Publishers/Pergamon Press, Beijing/Oxford, 1991.